

Alterations

Brian Conrad (Stanford University, Fall 2025)

notes by Aaryan Sukhadia, Miguel Machado, Rawley Harrison

Contents

1	Introduction to Resolution of Singularities	5
1.1	Some basic definitions	5
1.2	Some deep results	7
2	Using Alterations	10
2.1	Some applications and non-applications	10
2.2	Regularity versus Smooth	12
3	Semi-stability and Excellence	14
3.1	Basic Idea of deJong's proof	14
3.2	Semistable Curves	14
3.3	Excellence	17
4	Excellence II	19
4.1	Definition and examples	19
4.2	Relation between resolution of singularities and G-rings	21
5	Preliminary reduction steps	23
5.1	A note on modifications	23
5.2	Reduction steps	23
6	Curve Fibration I	27
6.1	Controlling the finite-fiber locus and enlarging Z	27
6.2	A model case of curve fibration via blow-up	28
7	Curve Fibrations II	31
7.1	Projection away from a well-chosen point	31
7.2	Bertini theorems and an application	32
8	Bertini and Birational Projections	34
8.1	Incidence correspondence: wrap-up for $d = N - 1$	34
8.2	Application of higher-dimensional Bertini	34

9	Curve Fibrations III	37
9.1	Finding the right blow-up locus	37
9.2	Relating fibral smoothness to relative smoothness	39
9.3	Genericity to ensure many smooth fibers	41
10	Stein factorization and the 3-point lemma	43
10.1	Geometric connectedness of fibers	43
10.2	Axiomatized setup and 3-point lemma	44
11	Finiteness aspect of 3-point lemma	47
11.1	Interpreting finiteness via linear spans	47
11.2	Passing to a high tensor power	48
12	Finer structure of D and Z	49
12.1	Completion of proof of 3-point lemma	49
12.2	Effect of alteration on Y	51
12.3	Arranging Z to be a union of sections	53
13	Stable Marked Curves	54
13.1	Definition and constructions	54
13.2	Gluing	54
13.3	Relative ingredients	58
14	Stability and Ampleness	60
14.1	Characterizations of stability	60
14.2	Relative ampleness and openness	61
15	Moduli of stable curves	65
15.1	Universal rigidified family over a Hilbert scheme	65
15.2	The moduli stack $\overline{\mathcal{M}}_{g,n}$	66
16	Some generalities on stacks	70
16.1	What is an Artin or Deligne-Mumford stack?	70
16.2	Properties of Artin stacks over a base scheme	73
17	Smoothness and properness of $\overline{\mathcal{M}}_{g,n}$	77
17.1	Base of induction	77
17.2	Stablization and contraction	81
18	Alteration to a stable curve I	86
18.1	Smoothness of $\overline{\mathcal{M}}_{g,n}$	86
18.2	Extending a rational map to $\overline{\mathcal{M}}_{g,n}$ via alteration	87
18.3	Flatness of graph closure after a modification	89
19	Normality and applications	92
19.1	Normality of semistable curves	92
19.2	Application to construction of $\mathcal{C} \rightarrow X$	94

20	Alteration to a stable curve II	97
20.1	Review of situation	97
20.2	Proof of condition (*)	97
20.3	Induction on the base	99
21	Artin approximation	100
21.1	Motivation via semistable curves	100
21.2	Henselian rings and local-étale maps	101
21.3	Main result and application	103
22	Ordinary Double Point Singularities	107
22.1	Definitions and examples	107
22.2	Structure Theorem	110
23	Refined structure of semistable singularities	116
23.1	Descending below the henselization	116
23.2	Measure of irregularity over a dvr	119
24	Blow-up for semistable curves over a dvr I	122
24.1	Overview about general blow-up and base change	122
24.2	Two charts on blow-up of C_n	125
25	Blow-up for semistable curves over a dvr II	127
25.1	All charts on blow-up of C_n	127
25.2	Scheme structure on the non-smooth locus	130
25.3	Return to main program	131
26	Local structure along codimension-2 singularities	133
26.1	Setup and main result	133
26.2	Étale-local structure and completions	134
27	Geometry for codimension-2 singularities	137
27.1	Description after completion	137
27.2	Consequences prior to completion	138
28	Semistability of a blow-up	141
28.1	Passage to a model case	141
28.2	Explicit blow-up calculation	144
29	Irreducible components of $\text{sing}(X)$	146
29.1	Geometry of irreducible components	146
29.2	Interaction of Z with modifications of X	148
30	Analyzing Z and resolving X	149
30.1	Structure of $Z \cap X^{\text{sm}}$	149
30.2	New axioms and final blow-ups	151
A	Blow-up review	153

A.1	Basic construction	153
A.2	Computing blow-ups	156
A.3	Strict transform	160
A.4	Functoriality and base change for blow-ups	162
A.5	Forcing disjointness via blow-ups	163
B	Iterated Genericity	165
B.1	Introduction	165
B.2	Genericity of L	167
B.3	Genericity of H	170
C	Hilbert schemes and a “universal” stable marked curve	172
C.1	Background on Hilbert schemes	172
C.2	Putting a stable marked curve into standard form	177
C.3	Hilbert scheme considerations	178
D	Isomorphisms of quadratic spaces	182
D.1	Introduction	182
D.2	Smoothness of an Isom-scheme	183
E	From normal crossings to strict normal crossings	186
E.1	Introduction	186
E.2	Construction of some closed sets	186
E.3	The blow-up process	188
F	Resolving singularities in codimension 3	190
F.1	Axiomatic setup	190
F.2	Blow-up in codimension 3	191
	References	194

Lecture 1

Introduction to Resolution of Singularities

Since time immemorial mathematicians have been looking for ways to get rid of singularities. This course will look at a specific technique to do so pioneered by deJong, and build up some of the theory along the way to understand the proof of his main result.

§1.1 Some basic definitions

DEFINITION 1.1.1. Let X be a reduced locally noetherian scheme. A **resolution of singularities** of X is a map of schemes $f : X' \rightarrow X$ such that:

- (1) f is proper.
- (2) X' is regular.
- (3) f is birational (i.e. there exist dense opens $U \subset X$, $U' \subset X'$ such that $f : U' \xrightarrow{\sim} U$).

EXERCISE 1.1.A. Show that condition (3) in the above definition is equivalent to the following: There exists a dense open $U \subset X$ such that $f^{-1}(U) \subset X'$ dense and $f^{-1}(U) \xrightarrow{\sim} U$.

REMARK 1.1.2. The above definition is given in extreme generality. In virtually all contexts within this course one could just as well replace the “reduced locally noetherian” in the above definition with “integral noetherian” if one so wishes.

We could be greedy in how nice we want a resolution of singularities to be. As motivation, we recall that virtually all organically occurring commutative rings in number theory or algebraic geometry are **excellent**¹, e.g. any complete noetherian local ring, any Dedekind domain of characteristic 0, rings of real or complex convergent power series, and any localizations or finitely generated algebras over the previous are all excellent. For any excellent scheme – which is a rather mild condition – the **regular locus**

$$\mathrm{Reg}(X) := \{x \in X : \mathcal{O}_x \text{ regular}\}$$

is (Zariski) open.

HOPE. The open set U along which the isomorphism $f : f^{-1}(U) \xrightarrow{\sim} U$ holds for a resolution of singularities can be arranged to be exactly the regular locus $\mathrm{Reg}(X)$. Moreover we could also ask

¹this notion will be discussed a bit later on in the course

that the closed set $X' - f^{-1}(U)$ (given the reduced structure) to be a “nice” closed subset of X' in the sense of the following definition.

DEFINITION 1.1.3. Let S be a regular scheme and let $D \subset S$ be an effective Cartier divisor (i.e. a closed subscheme whose ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_S$ is invertible). We say D is a **strict normal crossings divisor** (henceforth abbreviated to **sncd**) if:

- (1) D is reduced.
- (2) All the irreducible components $\{D_i\}_{i \in I}$ of D (given the reduced structure) are regular.
- (3) For any finite subset $J \subset I$ (the index set of the irreducible components of D), the intersection

$$D_J := \bigcap_{j \in J} D_j$$

is regular of pure codimension $\#J$ in S (i.e. $\dim \mathcal{O}_{D_J, \xi} = \dim \mathcal{O}_{S, \xi} - \#J$ for all $\xi \in D_J$).

The prototypical intuitive geometric picture to keep in mind for the above definition is a union of hyperplanes through the origin in an affine space; all the irreducible components are geometrically nice and taking successive intersections among such components decreases the dimension by 1 each time.

There are different ways one can characterize sncd's. For every $\xi \in D$, $(\mathcal{I}_D)_\xi = (t)$ is an ideal in the regular local (and thus UFD) $\mathcal{O}_{S, \xi}$, and $t = \prod_{i \in J \subset I} t_i$ where $(t_i) = (\mathcal{I}_{D_i})_\xi$ for D_i the irreducible component of D through ξ .

EXERCISE 1.1.B. Check that a reduced Cartier divisor $D \subset S$ is an sncd if and only if the irreducible factors of the local generators of $(\mathcal{I}_D)_\xi$ constitute part of a regular system of parameters in $\mathcal{O}_{S, \xi}$.

One difficulty with the notion of an sncd is that it is not an étale-local condition: one cannot determine whether a reduced closed subscheme in a regular scheme S is an sncd solely by looking at its pullback to some étale cover of S . Since the right notion of “stalk” in the étale topology is strict henselization, the difficulty boils down to the fact that the henselization of a local noetherian domain need not be a domain (much as for completion, as will be illustrated shortly).

EXAMPLE 1.1.4 (Failure to be an sncd). Consider the nodal cubic

$$C := \{y^2 = x^2(x + 1)\} \subset \mathbf{A}_k^2$$

for k a field of characteristic that is not 2. This is integral but not regular (this fails at the

origin $(0, 0)$), so it is not an sncd. The element $1 + x$ is a square in the completion

$$R := \widehat{\mathcal{O}}_{C,(0,0)} = k[[x, y]]/(y^2 - x^2(x + 1))$$

since the Taylor series of $\sqrt{1 + x}$ has coefficients which are all rational of denominator a power of 2. Thus that R is *not* a domain since we have

$$y^2 - x^2(x + 1) = (y - x\sqrt{1 + x})(y + x\sqrt{1 + x}),$$

yielding surjective maps $R \twoheadrightarrow k[[x, y]]/(y - x\sqrt{1 + x}) \simeq k[[x]]$ and $R \twoheadrightarrow k[[x, y]]/(y + x\sqrt{1 + x}) \simeq k[[x]]$. Similarly C has reducible pullback C' over the étale neighborhood

$$U := \operatorname{Spec}(k[x, y, \sqrt{1 + x}]_{(1+x)}) \rightarrow \operatorname{Spec}(k[x, y]_{(1+x)}) = \{x \neq -1\} \subset \mathbf{A}_k^2$$

of the origin, with C' an sncd in U . So although the geometry of C at the origin causes C not to be an sncd, it becomes an sncd over an étale neighborhood or “formal-locally” at the origin.

Motivated by the above example, we wish to weaken the notion of an sncd to allow for an étale-local notion:

DEFINITION 1.1.5. Let S be a regular scheme. A reduced effective Cartier divisor $D \subset S$ is a **normal crossings divisor (ncd)** if for all points $\xi \in D$, there exists an étale neighborhood $f : U' \rightarrow S$ such that $f^{-1}(D) \subset U'$ is an sncd.

For regular S that is excellent (a mild condition: see §3.3), the Artin-Popescu approximation theorem to be discussed later (§21.3) gives that a Cartier divisor $D \subset S$ is an ncd if and only if $\operatorname{Spec}(\widehat{\mathcal{O}}_{D,\xi}) \subset \operatorname{Spec}(\widehat{\mathcal{O}}_{S,\xi})$ is an sncd for all $\xi \in D$; we will not use this.

EXERCISE 1.1.C. Let S be a regular scheme. If a Cartier divisor $D \subset S$ is an ncd, show that $D \subset S$ is an sncd if and only if all irreducible components $D_i \subset D$ (with the reduced structure) are regular. The only non-trivial part here is the intersection codimension condition; one must traverse between D and its pre-image under an étale map to verify this.

§1.2 Some deep results

The dream scenario is to have a resolution of singularities that is an isomorphism over the regular locus of a given scheme and has preimage in the resolution whose complement is an ncd. The following result has been described by Kollár as Hironaka’s magnum opus.

THEOREM 1.2.1 (Hironaka). For a reduced separated scheme X of finite type over a field k of characteristic 0, there exists a resolution of singularities $f : X' \rightarrow X$ such that

1. f is an isomorphism over $\text{Reg}(X) = X^{\text{sm}} \subset X$, and
2. $f^{-1}(X - \text{Reg}(X)) \subset X'$ is an ncd.

In characteristic $p > 0$ the analogous statement remains unknown (but it is known when $\dim X \leq 2$).

REMARK 1.2.2 (Historical Aside). The original formulation of the theorem was in greater generality and its original proof is notoriously complex, so much so that Deligne found it overwhelming to navigate the many intricacies of Hironaka's paper (split into two parts, one 95 pages [Hir64a] and the other 122 pages [Hir64b]) after taking it with him on his year in the Belgian army. The proof has been revised and simplified by many people over the years; Kollár has a rather efficient 30(ish)-page proof in [Kol07].

Johan deJong took a different approach to the problem of resolving singularities by introducing the weaker notion of an **alteration** that turns out to often be sufficient and permits a striking existence resolution in positive characteristic as well.

DEFINITION 1.2.3. Let X be an integral Noetherian scheme. An **alteration** of X is a map $f : X' \rightarrow X$ where

1. X' is integral and regular,
2. f is dominant proper (and thus surjective),
3. $[k(X') : k(X)] < \infty$.

EXERCISE 1.2.A. Use a standard spreading out argument to show condition (3) in the definition above is equivalent to the following; there exists a dense open $U \subset X$ such that $f^{-1}(U) \rightarrow U$ is finite flat (even finite étale when $k(X')/k(X)$ is separable).

The 3rd condition in Definition 1.2.3 makes the geometry of X' rather different from that of X ; e.g., *any* regular curve C over a field k a function field which is a finite extension of $k(t)$, so if C is proper then it is a finite cover of $\mathbf{P}_k^1$. So in some sense an alteration loses a huge amount of geometry, but for various uses this turns out not to be as detrimental as one might imagine. The main aim of this course is to prove:

THEOREM 1.2.4 (deJong). For k any field, let X be a k -variety (integral separated of finite type) and pick any proper closed subset $Z \subset X$. Then we have a diagram

$$\begin{array}{ccccc} f^{-1}(Z) & \hookrightarrow & X' & \xhookrightarrow{\iota} & \overline{X'} \\ \downarrow f & & \downarrow f & & \\ Z & \hookrightarrow & X & & \end{array}$$

where $f : X' \rightarrow X$ is an alteration, $\overline{X'}$ is a regular projective k -variety and $\iota : X' \hookrightarrow \overline{X'}$ is a dense open immersion such that the closed set $f^{-1}(Z) \cup (\overline{X'} - X') = \overline{X'} - f^{-1}(X - Z)$ with the reduced structure is an sned in $\overline{X'}$.

WARNING. The definition of variety in the above theorem, which we shall use as in deJong's paper, does not require geometric integrality (oft stipulated by some).

- REMARK 1.2.5.**
1. The theorem will ensure that if k is perfect then we can arrange $k(X')/k(X)$ to be separable (equivalently $X' \rightarrow X$ is finite étale over a dense open in X).
 2. The proof of this theorem gives *no* control on how $\overline{X'} - Z$ relates to $\text{Reg}(X)$. In particular it does not imply Theorem 1.2.1 (as far as is known), nor does the method enable one to ensure $X' \rightarrow X$ is étale or even quasi-finite over a chosen regular point in X . Nonetheless, deJong's method was adapted by Abramovich and deJong to prove a weaker version of Hironaka's theorem in characteristic 0 with birational maps (not just an alteration) [Ad97]. We will discuss next time with some examples that for certain types of statements the existence of a regular alteration can be used in place of appeals to Hironaka's great theorem.

Lecture 2

Using Alterations

§2.1 Some applications and non-applications

Hironaka's theorem was so optimal and versatile that for several decades it blinded everyone to weaker alternatives that might be within reach in positive characteristic (though a small community did work on finding shorter approaches to the proof of Hironaka's result). In this section we go over some applications of the theorem in which alterations in place of birational maps actually suffices.

The Riemann Existence Theorem. Riemann proved the algebraicity of compact Riemann surfaces by showing (in modern language) that any compact connected Riemann surface X admitting a non-constant meromorphic function (which thereby exhibits X as a finite flat cover of $\mathbf{CP}^1$). There are various ways to generalize this in the language of algebraic varieties or monodromy. The following vast generalization, called “Riemann Existence Theorem” in [SGA1, Exp. XII, Thm. 5.1], builds on a key ingredient due to Grauert and Remmert.

THEOREM 2.1.1. Let X be a locally finite-type $\mathbf{C}$ -scheme, and let $X \mapsto X^{\text{an}}$ be the analytification functor from the category of such schemes to the category of complex-analytic spaces. For the categories

$$F_{\text{ét}}(X) = \{\text{finite étale covers } E \rightarrow X\}$$

and

$$F_{\text{ét}}(X^{\text{an}}) := \{\text{finite étale covers } E' \rightarrow X^{\text{an}}\} = \{\text{finite-degree covering spaces } T \rightarrow X(\mathbf{C})\},$$

the analytification functor $E \mapsto E^{\text{an}}$ is an equivalence of categories.

If X is assumed to be proper then the above theorem is a relatively simple consequence of Serre's GAGA. Grothendieck's proof in the general case involves working locally on X to reduce to affine X that is made reduced, then normal, and Hironaka's resolution of singularities is applied to the affine normal X embedded in a projective closure. The use of Hironaka's result can be replaced with deJong's alterations.

EXERCISE 2.1.A. Adapt the proof of Theorem 2.1.1 in [SGA1, Exp. XII] to use deJong's theorem on alterations in place of Hironaka's result.

Artin Comparison Theorem. Suppose $\mathcal{F}$ is an abelian sheaf on the étale site of a locally finite type $\mathbf{C}$ -scheme X . The analytification functor (used in Theorem 2.1.1) gives rise to an analogous functor at the level of abelian sheaves to produce $\mathcal{F}^{\text{an}}$ on $X(\mathbf{C})$. This can a powerful tool for computing étale cohomology, such as in the following result.

THEOREM 2.1.2 (Artin Comparison). Let X be a separated finite-type $\mathbf{C}$ -scheme, and $\mathcal{F}$ a constructible abelian étale sheaf on X . The natural map

$$H^\bullet(X_{\text{ét}}, \mathcal{F}) \rightarrow H^\bullet(X(\mathbf{C}), \mathcal{F}^{\text{an}})$$

is an isomorphism

The analogous statement for $H_c^\bullet$ (cohomology with compact support) is significantly easier to prove via an inductive argument with excision sequences, base change theorems, and curve fibrations to ultimately reduce to the case of curves. Such inductive tools are not available for $H^\bullet$, which makes the proof of the Artin comparison theorem for it far more difficult. The proof in [SGA4] uses resolution of singularities. Later Berkovich [Ber93, Thm. 7.5.1(iii)] gave an entirely different approach for proving an analogous comparison theorem with étale cohomology for non-archimedean analytic spaces (over any non-archimedean field, which could have *positive* characteristic) in which all need for resolution-type results was eliminated. Berkovich’s method adapts to make sense over $\mathbf{C}$, where it removes all use of resolution-type tools in the proof of Theorem 2.1.2.

However, there remains an important role for resolution in this matter: bootstrapping to constructible ℓ -adic sheaves $\mathcal{F}$ with arbitrary singularities for X . This upgrading to the ℓ -adic case is not too hard to prove in the smooth case, but in general it uses the local triangulizability of $X(\mathbf{C})$. Happily, that triangulation result can be bypassed: an elegant argument of Deligne (private communication) uses alterations to carry out a slick reduction to the smooth case.

Mixed Hodge Structures. In Deligne’s theory of mixed Hodge structures for singular algebraic schemes X over $\mathbf{C}$, he used Hironaka’s resolution to build “proper hypercoverings” of X whose layers are smooth. One can use alterations for the same purpose, and the rest of Deligne’s development goes through without any changes.

Algebraic deRham theory (a non-application). Let X be a *smooth* separated scheme of finite type over $\mathbf{C}$. We have the algebraic deRham complex (*not* exact for the Zariski topology)

$$\Omega_{X/\mathbf{C}}^\bullet = (\mathcal{O}_X \xrightarrow{d} \Omega_{X/\mathbf{C}}^1 \xrightarrow{d} \Omega_{X/\mathbf{C}}^2 \xrightarrow{d} \cdots)$$

which gives the deRham cohomology $H_{\text{dR}}^i(X/\mathbf{C}) := \mathbf{H}^i(X, \Omega_{X/\mathbf{C}}^\bullet)$, where $\mathbf{H}^\bullet(-)$ denotes the hypercohomology of a complex. A natural analytification map

$$H_{\text{dR}}^i(X/\mathbf{C}) \rightarrow \mathbf{H}^i(X^{\text{an}}, \Omega_{X^{\text{an}}/\mathbf{C}}^\bullet) \simeq H^i(X(\mathbf{C}), \mathbf{C})$$

(last step via the complex-analytic Poincaré Lemma) is an isomorphism [Gro66, Thm. 1’].

REMARK 2.1.3. For proper X this “deRham comparison map” can be shown to be an isomorphism by use the Leray spectral sequence to reduce to the GAGA comparison isomorphisms for $H^i(\Omega^j)$.

The proof that $H_{\text{dR}}^i(X/\mathbf{C}) \rightarrow H^i(X(\mathbf{C}), \mathbf{C})$ is an isomorphism first uses a Čech type reduction to quasi-projective X , and then Hironaka’s theorem (applied to a projective closure of X) yields an open immersion $X \hookrightarrow \overline{X}$ with smooth projective $\overline{X}$ for which $\overline{X} - X$ is an ncd inside $\overline{X}$. Grothendieck exploited the favorable geometry of ncd’s to then deduce the desired isomorphism property for X from the settled proper case. Since $H_{\text{dR}}^i(X/\mathbf{C})$ behaves poorly with respect to proper surjections, and deJong’s theorem also doesn’t control the quasi-finite or étale locus in the base, it doesn’t seem possible to replace this use of Hironaka’s theorem with deJong’s.

§2.2 Regularity versus Smooth

Let S be a scheme locally of finite type over a field k . We say S is *regular* if $\mathcal{O}_{S,s}$ is regular for all $s \in S$. (It is sufficient to check regularity of $\mathcal{O}_{S,s}$ for closed $s \in S$, by Serre’s result that the localization of a regular local ring at a prime ideal is regular.) Note that this makes no reference to the k -structure on S ; it is a property of the abstract scheme S .

We say that S is *smooth over k* if any of the following (not obviously!) equivalent conditions are satisfied (writing S_K to denote $S \otimes_k K$ for a field extension K/k):

1. $S_{\overline{k}}$ is regular,
2. $S_{k'}$ is regular for all finite k'/k ,
3. $S_{k'}$ is regular for all k'/k ,
4. $S_{k'}$ is regular for some perfect k'/k ,
5. $S \rightarrow \text{Spec}(k)$ satisfies the infinitesimal smoothness criterion.

When k is perfect then smoothness over k is equivalent to regularity and otherwise these two conditions are distinct:

EXAMPLE 2.2.1. Let k be an imperfect field with $\text{char } k = p > 0$ and $a \in k - k^p$ (e.g. $k = \mathbf{F}_p(t)$ and $a = t$). Pick $m > 1$ with $p \nmid m$. Let $C = \{y^m = x^p - a\} \subset \mathbf{A}_k^2$. Since $a \notin k^p$, $x^p - a$ is irreducible in $k[x]$ and one checks using the ramification theory of dvr’s that $k[x, y]/(y^m - (x^p - a))$ is Dedekind (showing it is the integral closure of $k[x]$ in the degree- m extension $k(x)((x^p - a)^{1/m})$.) In particular C is regular.

Over $\overline{k}$ the defining equation becomes $y^m - (x^p - a) = y^m - (x - \alpha)^p$ for $\alpha^p = a$. Therefore,

$C_{\bar{k}}$ is integral (as $p \nmid m$) but it has a unique singularity at $(\alpha, 0)$. Clearly $C - \{\xi\}$ is k -smooth for the closed point $\xi = (x^p - a, y)$.

The preceding example is k -smooth on a dense open. More broadly, such generic smoothness for a curve encodes a property of the field extension $k(C)/k$ as we explain with the next class of examples.

EXAMPLE 2.2.2. Let K/k be finitely generated of transcendence degree 1; using normalization of a projective model we have $K = k(C)$ for a unique regular proper connected curve C over k . Choices of finite $k(x) \rightarrow K$ correspond to finite flat maps $C \rightarrow \mathbf{P}_k^1$ over k . If C has a k -smooth dense open subset U then there exists an étale k -map $U \rightarrow \mathbf{A}_k^1$ where $x \in k$ corresponds to separating transcendence basis over k for $k(U) = k(C) = K$. In particular K/k having a separating transcendence basis (equivalent to K/k being separable in the sense of field theory [Mat90, §26]) is equivalent to the existence of a k -smooth dense open $U \subset C$.

The next example has the remarkable feature of being regular but *not* generically k -smooth (in fact it is geometrically *nowhere reduced*) yet with k algebraically closed in the function field (so the presence of nilpotents over $\bar{k}$ is not caused by the presence of a nontrivial purely inseparable extension of the constant field occurring inside the function field).

EXAMPLE 2.2.3 (MacLane). Let $k = \mathbf{F}_p(s, t)$ and $C = \{sx^p + ty^p = 1\} \subset \mathbf{A}_k^2$. It is readily verified that C is integral and with a bit more work that C is Dedekind, and more challenging to prove (exercise!) that k is algebraically closed in $k(C)$.

Since $C_{\bar{k}} = \{(\sigma x + \tau y)^p = 1\}$ for $\sigma^p = s$ and $\tau^p = t$, we see that C is geometrically irreducible but geometrically everywhere non-reduced. In particular, $k(C)/k$ does not have a separating transcendence basis since C does not have a k -smooth dense open subset.

More broadly, these considerations show that if X is a variety over k (in the sense of [deJ96]: an integral separated scheme of finite type over k) then even if k is algebraically closed in $k(X)$ there may *not* exist a k -smooth alteration. Indeed, if $k(X)/k$ is *not* separable (i.e. no separating transcendence basis) then there is no k -smooth alteration $X' \rightarrow X$ since otherwise the composite field inclusion $k \subset k(x) \subset k(X')$ being separable would imply $k(X)/k$ is separable (use the characterizations of separability in [Mat90, Thm. 26.2]).

Thus, the best one can hope for in general is the existence of a regular alteration rather than a k -smooth one (if k is not perfect), even if k is algebraically closed in $k(X)$. When X is generically k -smooth (and k is not perfect) then deJong's proof still produces only a regular alteration rather than a k -smooth one.

Lecture 3

Semi-stability and Excellence

§3.1 Basic Idea of deJong's proof

Consider a proper closed subset $Z \subsetneq X$ with X a variety over k .

Step 0: Use Chow's Lemma and normalization to reduce to X quasi-projective and normal, and then to the case $k = \bar{k}$. This settles dimension $d = 1$ ($d = 0$ is trivial), and we will induct on $d > 1$.

Step 1: Use blow-ups to pass to where there is a surjective map $X \twoheadrightarrow \mathbf{P}^{d-1}$ with fibers smooth and geometrically connected curves over a dense open.

Step 2: Use properness of the moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable genus- g curves with n marked points (which we'll discuss later) for suitable n to build alteration $Y' \rightarrow \mathbf{P}_k^{d-1}$ and $X' \rightarrow X$ fitting into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\text{alteration}} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & \mathbf{P}^{d-1} \end{array}$$

where $X' \rightarrow Y'$ has all fibers as semistable geometrically connected curves (defined below); beware that Y' may be awful.

This is where alterations most essentially arise, due to their role in the “semistable reduction theorem for curves” (to be discussed a bit later in this lecture) that in turn underlies properness of the global stack $\overline{\mathcal{M}}_{g,n}$.

Step 3: Apply induction to Y' (and its proper closed locus Z' of non-smooth fibers for $X' \rightarrow Y'$) to pass to where Y is smooth and the family $X \rightarrow Y$ of geometrically connected semistable curves is smooth over $Y - \tilde{Z}$ for an sncd $\tilde{Z} \subset Y$. This is now a very favorable geometric setup: a smooth base Y and a curve fibration $X \rightarrow Y$ with only mild singularities that moreover are supported over an sncd in Y . Concrete blow-ups on X (no more alterations) then conclude the proof.

§3.2 Semistable Curves

We begin our discussion of semistability for curves with the theory over a field.

THEOREM 3.2.1. Let X be a pure 1-dimensional finite type scheme over k . Pick $x \in X$ closed. The following are equivalent:

- (1) For some $\bar{x} \in X_{\bar{k}}$ over x , $\mathcal{O}_{X_{\bar{k}, \bar{x}}}^\wedge$ is $\bar{k}$ -isomorphic to either $\bar{k}[[t]]$ or $\bar{k}[[u, v]]/(uv)$

- (2) Either x lies in the open k -smooth locus X^{sm} or there exists a common étale neighborhood (over k)

$$\begin{array}{ccc} & (U, u) & \\ \text{étale} \swarrow & & \searrow \text{étale} \\ (X, x) & & (\{uv = 0\}, (0, 0)) \end{array}$$

(which implies that X is k -smooth on $V - \{x\}$ for some open $V \ni x$).

When these equivalent conditions hold, we say X is *semi-stable* at x .

Proof. The difficult part $(1) \Rightarrow (2)$ uses Artin approximation (as a special case of the study of “ordinary double point” singularities carried out in [FK88, §2, Ch. III]). $\square$

Note the consequence that if X is semistable at x which is not a k -smooth point then $k(x)/k$ is *separable* (thanks to the description using a common étale neighborhood with the origin in $\{uv = 0\}$). This may seem like a minor observation but is technically extremely useful later on.

DEFINITION 3.2.2. We say X is **semi-stable** if it is semi-stable at all closed points $x \in X$, in which case the open subset $X^{\text{sm}} \subset X$ is a dense open.

Beware that we do *not* require semistable curves (which are clearly k -smooth on a dense open) to be irreducible. This is very important for stating the “semistable reduction theorem for curves” (which in turn underlies properness of associated moduli stacks of stable curves), as already for elliptic curves over the fraction field of a dvr R a good notion of “semistable reduction” involves a proper flat R -model whose special fiber tends to be reducible.

To explain the general “semistable reduction theorem” for curves, we start with the following result for abelian varieties elegantly proved by Deligne at the end of [SGA7, Exp. I]:

THEOREM 3.2.3 (Semi-stable reduction of abelian varieties). Let A be an abelian variety over $K = \text{Frac}(R)$ for a dvr R . There exists a finite separable extension K'/K so for the R -finite semi-local integral closure $R' \subset K'$ of R the Néron model $\mathcal{A}' = \text{Néron}(A_{K'})$ has “semistable reduction”: the identity component of each geometric special fiber of $\mathcal{A}'$ has “affine part” that is a torus.

Here, “affine part” refers to a general theorem of Chevalley [Con02] that every smooth connected group variety G over a perfect (e.g., algebraically closed) field is uniquely an extension of an abelian variety by a smooth connected affine group G^{aff} . The requirement for the affine part to be a torus is illuminated partly by the general fact any *commutative* smooth connected affine group over a perfect field is uniquely the product of a torus and a commutative unipotent group.

REMARK 3.2.4. Letting k be the residue field of R , in Theorem 3.2.3 is enough (by Deligne’s proof) to take K'/K that splits the finite étale K -group $A[\ell]$ for an odd prime $\ell \neq \text{char}(k)$ or $\ell = 4$ (when $\text{char}(k) \neq 2$).

To apply the preceding theorem to curves, consider smooth proper and geometrically connected curve X over $K = \text{Frac}(R)$. There exists a “minimal regular proper model” of X : a proper flat morphism $\mathcal{X} \rightarrow \text{Spec}(R)$ with generic fiber X such that $\mathcal{X}$ is regular and has “no contractions” in its special fiber (minimality): this means no proper birational $f : \mathcal{X} \rightarrow \mathcal{Y}$ to another *regular* proper flat R -scheme $\mathcal{Y}$ with f not sending an irreducible component of $\mathcal{X}_k$ onto a point (this is equivalent to f not being quasi-finite, which in turn is equivalent to it not being an isomorphism by normality considerations). A minimal regular proper model is unique when $g > 0$.

The existence of a minimal regular proper model is a combination of two deep ingredients: Lipman’s resolution of singularities for abstract excellent (§3.3) surfaces and the Lichtenbaum/Shafarevich theory of minimal models for regular proper flat R -curves. These ingredients are developed in [CS86, Ch. XI, XII] (with a minor restriction on residue fields for abstract surface resolution in equicharacteristic $p > 0$ that $[k(x)^p : k(x)] < \infty$ for all closed points x , which will be harmless for our later purposes with proper flat curves with smooth generic fiber over a dvr since we will only need it when the residue field is algebraically closed; alternatively, [SP, Tag 0ADW] gives a proof without this restriction, the key points being [SP, Tag 0B4U] and its use in [SP, Tag 0BGN]). Although there are non-excellent dvr’s (to be discussed next time), $\hat{R}$ is always excellent and this can be used via a trick to make a regular proper flat model over a general dvr R when one is in hand over $\hat{R}$.

The geometry of a minimal regular proper model can be connected to the geometry of the Néron model of the Jacobian $A := \text{Pic}_{X/K}^0$ (an abelian variety with dimension $g := \text{genus}(X)$). This a fundamental discovery of Deligne and Mumford [DM69, Prop. 2.3, Thm. 2.4]:

THEOREM 3.2.5 (Deligne-Mumford). If $g \geq 2$ and the residue field k of R is perfect then A has semistable reduction if and only if the minimal regular proper model $\mathcal{X} \rightarrow \text{Spec}(R)$ has semistable special fiber.

The formation of both the minimal regular proper model (in positive genus) and the Néron model commute with base change to the strict henselization R^{sh} , so for the purposes of the proof one can assume k is algebraically closed (as is done in the proof by Deligne and Mumford). The hypothesis of perfect residue field is not an obstacle for the purpose of proving a “semistable reduction theorem for curves” as stated without restriction on k below. The reason is that there is always a *local* dvr extension $R \rightarrow R'$ with ramification degree 1 inducing a separable algebraic extension K'/K on fraction fields and the perfect closure k_p/k on residue fields (proved by a Zorn-style argument inside K_s , using that a finite purely inseparable extension of the residue field can be realized with a

ramification-degree 1 *separable* extension of the fraction field), and we can use the desired conclusion over such an R' to get what we want over a finite separable extension of the original K .

What about $g \leq 1$? Any smooth curve (as for any non-empty smooth scheme over a field) acquires a rational point over a finite separable extension of the base field, so for $g = 0$ it becomes $\mathbf{P}^1$ over a finite separable extension of the base field and for $g = 1$ it becomes an elliptic curve (i.e., a 1-dimensional abelian variety) over such an extension of the base field. For elliptic curves, the Néron model has semistable reduction if and only if the minimal Weierstrass has semistable special fiber [Con15, Thm. 5.5], in which case (by the theory of relative semistable curves over dvr 's to be discussed later on) a regular resolution constructed by explicit blow-up is seen by étale-local calculation to be a minimal regular model with semistable special fiber

Combining these special facts for $g \leq 1$ with Theorem 3.2.5 for $g \geq 2$ and Theorem 3.2.3 for abelian varieties yields:

THEOREM 3.2.6 (Semistable reduction theorem for curves). For a smooth proper geometrically connected curve X over $K = \text{Frac}(R)$, there exists a finite separable K'/K and proper flat R' -curve $\mathcal{X}'$ with semistable special fibers and generic fiber K' -isomorphic to $X_{K'} := X \otimes_K K'$.

The reliance upon Néron models of abelian varieties for this result about curves might be bothersome. An extremely combinatorial approach to semistable reduction for curves that still uses abstract surface resolution but no abelian varieties was worked out by Artin and Winters and is presented in [SP, Tag 0C2Q].

§3.3 Excellence

It is technically very convenient to reduce some tasks to calculations with *completed* local rings, essentially because completions may admit more explicit descriptions than local rings for the Zariski topology. As a basic example, the algebraic local rings at closed points on a smooth curve over an algebraically closed field k cannot be described in any general explicit way but their completions are always the same: $k[[t]]$. But when passing between a local ring $A = \mathcal{O}_{X,x}$ on a variety and its completion $\hat{A}$, what properties are preserved or possibly lost? This was a very active area of study in the mid-20th century, initially by Zariski and Nagata, and put into its best formulation with Grothendieck's theory of excellence.

The upshot is that most noetherian rings one encounters are excellent, and it guarantees many nice facts for passing between A and $\hat{A}$. We will harness the power of excellence in several places, and to that end we now take up a short summary about the concept. The two best references on this are [Mat80, Ch. 13] (refers to some earlier commutative algebra results also proved in [Mat90]) and [EGA, §5-§7, esp. §7.8ff]. We'll get to the main definition next time, and for now we introduce:

DEFINITION 3.3.1. A noetherian ring A is called **catenary** if for all $\mathfrak{p} \subsetneq \mathfrak{p}'$ in A , all maximal chains of primes between them (in the sense that there are no intermediary primes at each step, not maximal in length) chains $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}'$ have $n = \dim A_{\mathfrak{p}'} - \dim A_{\mathfrak{p}}$. We say A is **universally catenary** if all finitely generated A -algebras are catenary.

Note that this definition is really a condition on the domains $A/\mathfrak{p}$. Informally, catenary rings are those with a well-behaved theory of codimension. For instance, fields are universally catenary thanks to the good behavior of codimension for varieties over a field. It is difficult to build non-catenary noetherian rings, but they do exist.

EXAMPLE 3.3.2. ([Mat90, §17.4]) Every (quotient of) a Cohen-Macaulay ring is catenary. If a Cohen-Macaulay ring B maps onto A then $B[T_1, \dots, T_n] \twoheadrightarrow A[T_1, \dots, T_n]$ so such A are even universally catenary.

EXAMPLE 3.3.3. The Cohen Structure Theorem tells us that every complete local noetherian A has a surjection $B \twoheadrightarrow A$ with B regular (hence CM), so all such A are universally catenary.

Lecture 4

Excellence II

§4.1 Definition and examples

DEFINITION 4.1.1. A noetherian ring A is called **excellent** if it satisfies the following properties:

1. it is universally catenary,
2. it is a **G-ring**: for all primes $\mathfrak{p}$ of A , the faithfully flat map $\mathrm{Spec}(A_{\mathfrak{p}}^{\wedge}) \rightarrow \mathrm{Spec}(A_{\mathfrak{p}})$ has geometrically regular fibers (i.e., for every prime $\mathfrak{q} \subset \mathfrak{p}$ the noetherian “formal fiber algebra” $A_{\mathfrak{p}}^{\wedge} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{q})$ is regular and remains regular after all finite extensions on $\kappa(\mathfrak{q})$);
3. for all finitely generated A -algebras A' , the regular locus $\mathrm{Reg}(A') \subset \mathrm{Spec}(A')$ is open.

The conditions (2) and (3) together are called *quasi-excellent*. By far the most subtle of these conditions is the G-ring property (named after Grothendieck, who conceived of this daring condition under which one should expect interesting properties of $A_{\mathfrak{p}}^{\wedge}$ to descend to $A_{\mathfrak{p}}$ due to results on “flatness and fibers” as in [Mat90, §23], and more importantly *proved* the G-ring condition is very robust: inherited under passage to finitely generated algebras). It should also be noted that since condition (3) may seem hard to verify in practice, it does admit more readily verifiable equivalent formulations such as [Mat90, Thm. 24.4] (also see [EGA, IV₂, 6.12.4]).

EXAMPLE 4.1.2. Consider a dvr R with fraction field K . The condition for A to be a G-ring is that $K' \otimes_K \widehat{K}$ is reduced for all finite extensions K'/K , which is to say $\widehat{K}/k$ is a separable extension in the sense of field theory (equivalently, all subextensions of $\widehat{K}$ finitely generated over K admit a separating transcendence basis). This is automatic when $\mathrm{char}(K) = 0$, and in [BLR90, §3.6, Ex. 11] there is given an example of a non-excellent dvr due to failure of the G-ring property.

EXAMPLE 4.1.3. All fields, and all Dedekind domains with fraction field of characteristic 0 (such as $\mathbf{Z}!$), are G-rings.

EXERCISE 4.1.A. For a regular curve C over any field k , show that $K := k(C)$ has $\widehat{K}/K$ separable in the sense of field theory (i.e., $K' \otimes_K \widehat{K}$ is reduced for all finite extensions K'/K). Thus, the algebraic local rings $\mathcal{O}_{C,c}$ at closed points are G-rings and so are excellent (parts (1) and (3) of the definition of excellence are more readily verified; e.g., (3) can sometimes be studied using [Mat90, Thm. 24.4], which allows one to infer openness of the regular locus for finitely generated algebras over a field k by harnessing generic smoothness for reduced

algebras of finite type over $\bar{k}$ and inheritance of regularity from regularity of a flat cover [Mat90, Thm. 23.7(i)].

THEOREM 4.1.4 (Grothendieck/Nagata; EGA IV₂, §7.8). Excellence satisfies the following properties:

1. Every complete local noetherian ring and every Dedekind domain of characteristic 0 is excellent.
2. Excellence is inherited under passages to finitely generated algebras and localization at multiplicative sets (e.g., every localization of a finitely generated $\mathbf{Z}$ -algebra is excellent).
3. If A is excellent and reduced then the normalization map $A \rightarrow \hat{A}$ is module-finite.
4. If $\mathbf{P}$ is any of a long list of “nice” homological properties of local noetherian rings and

$$\mathbf{P}(A) := \{\mathfrak{p} \in \operatorname{Spec}(A) : A_{\mathfrak{p}} \text{ satisfies } \mathbf{P}\}$$

then

- $\mathbf{P}(A)$ is open in $\operatorname{Spec}(A)$,
- for all ideals $I \subset A$ and the I -adic completion $\hat{A}$ of A , the map $f : \operatorname{Spec}(\hat{A}) \rightarrow \operatorname{Spec}(A)$ satisfies $f^{-1}(\mathbf{P}(A)) = \mathbf{P}(\hat{A})$ (in particular, if A is local excellent and $\hat{A}$ is the completion at the maximal ideal then A satisfies $\mathbf{P}$ if and only if $\hat{A}$ does!)

Assertion (2) in the preceding result is the most remarkable aspect, in view of how the G-ring condition is defined.

DEFINITION 4.1.5. A locally noetherian *scheme* X is **excellent** if every affine open $\operatorname{Spec}(A) \subset X$ has A excellent, or equivalently that all members of some affine open cover have excellent coordinate rings.

REMARK 4.1.6. There is a “Jacobian criterion” for excellence of regular $\mathbf{Q}$ -algebras [Mat80, Thm. 101], essentially that there are as many linearly independent derivations as the dimension (in the local case). With this one can verify excellence of the analytic local ring of holomorphic functions $\mathcal{O}_{\mathbf{C}^n,0}^{\text{an}}$ and hence of the local ring on any complex-analytic space.

Since excellence allows one to pass back and forth between a local ring and its completion for various good homological properties, it follows that if X is any locally finite type $\mathbf{C}$ -scheme then properties transfer back and forth between $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X^{\text{an}},x}$ for all $x \in X(\mathbf{C})$ because these two excellent local noetherian rings naturally have the *same* completion (ultimately due

to the case of the origin in affine space).

A similar conclusion holds for algebraic schemes over non-archimedean fields k and the associated rigid-analytic spaces, but the proof is more complicated (e.g., we cannot use a Jacobian criterion for excellence when $\text{char}(k) > 0$). In general k -affinoid algebras and rigid-analytic local rings are excellent, but this is much harder than the work in the complex-analytic case.

§4.2 Relation between resolution of singularities and G-rings

The following discussion is never going to be used, but it is psychologically valuable for highlighting why the G-ring condition (and so ultimately quasi-excellence) is a natural concept in any general study of resolution of singularities.

THEOREM 4.2.1. [EGA, IV₂, 7.9.5] Let X be a locally noetherian scheme. Assume for all open $U \subset X$ and all finite $Y \rightarrow U$ with Y integral, there exists a resolution of singularities for Y . Then X is quasi-excellent (i.e., all local rings $\mathcal{O}_{X,x}$ are quasi-excellent).

The statement of this result in [EGA] omits mention of U (i.e., only finite $Y \rightarrow X$ with Y integral are considered), but this does not seem to be correct since such a global hypothesis does not seem to be “localizable” enough for the proof to work (why should a finite scheme over an open $U \subset X$ extend to a finite X -scheme?). Note also that we are not requiring $Y \rightarrow U$ to be surjective; it may be a finite cover of an irreducible component of U .

Proof. We focus on the G-ring aspect, which is where the real substance lies, so (thanks to our hypotheses being localizable) we may assume $X = \text{Spec}(A)$ is affine. We want to show for primes $\mathfrak{p}$ of A that the formal fiber algebras of $\text{Spec}(A_{\mathfrak{p}}^{\wedge}) \rightarrow \text{Spec}(A_{\mathfrak{p}})$ are geometrically regular. We omit some routine technical details in what follows (such as tracking how our hypotheses behave under some reduction steps), to focus on the key points.

Pick a prime $\mathfrak{q} \in \text{Spec}(A_{\mathfrak{p}})$; i.e., a prime of A inside $\mathfrak{p}$. We want to show that the $\kappa(\mathfrak{q})$ -algebra $A_{\mathfrak{p}}^{\wedge} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{q})$ is regular and remains as such after scalar extension to all finite extensions of $\kappa(\mathfrak{q})$. We may pass to the quotient of A by the prime corresponding to $\mathfrak{q}$ to reduce to the key case when A is a local domain; i.e., for $K = \text{Frac}(A)$ we want to show that $\hat{A} \otimes_A K$ is regular and remains regular after scalar extension to any finite extension of K .

For a finite extension K'/K , by denominator-chasing we have $K' = \text{Frac}(A')$ for an A -finite (hence semi-local) domain $A' \supset A$, and then

$$\hat{A} \otimes_A K' = (\hat{A} \otimes_A A') \otimes_{A'} K' = \prod_{\mathfrak{m}'} (A'_{\mathfrak{m}'})^{\wedge} \otimes_{A'_{\mathfrak{m}'}} K'$$

where $\mathfrak{m}'$ varies through the finitely many maximal ideals of A' . Upon renaming $A'_{\mathfrak{m}'}$ as A , we come down to the essential case $K' = K$. This amounts to the following:

Claim: *If A is a local noetherian domain with fraction field K and $X = \operatorname{Spec}(A)$ admits a resolution of singularities then $\hat{A} \otimes_A K$ is regular.*

To prove this Claim, if $f : Y \rightarrow X$ is a resolution of singularities (so Y is regular and irreducible and f is a proper map that is an isomorphism over a dense open subscheme of X) then for the natural faithfully flat map $h : \operatorname{Spec}(\hat{A}) \rightarrow \operatorname{Spec}(A)$ form the pullback diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\quad} & Y \\ f' \downarrow & & \downarrow f \\ X' = \operatorname{Spec}(\hat{A}) & \longrightarrow & \operatorname{Spec}(A) = X \end{array}$$

Our task is to show the generic fiber of h (a “localization” of X') is a regular scheme. All points in that generic fiber lie over the generic point of X and hence over a dense open $U \subset X$ over which f is an isomorphism, so correspondingly f' is an isomorphism over the open $h^{-1}(U)$ around all such points of X' . Hence, to proving regularity of the local rings of X' at all such points it suffices to prove Y' is a regular scheme.

Although $Y' \rightarrow Y$ is faithfully flat (a base change of the faithfully flat h) and its base Y is regular, we can't apply any off-the-shelf commutative algebra results to ascend regularity because *we don't yet know* that the fibers of $Y' \rightarrow Y$ are regular (as we don't know the fibers of h are geometrically regular; the G-ring property of A is *not* known and that is what we are trying to prove).

Now for the brilliant step. Since the complete local noetherian ring $\hat{A}$ is excellent, the regular locus of every finite type scheme over $\hat{A}$ is *open*. In particular, $\operatorname{Reg}(Y')$ is at least open in Y' . Since Y' is *proper* over the local X' , an open subset of Y' that contains the entire special fiber of $Y' \rightarrow X'$ must be the entire space. Hence, it suffices to prove Y' has regular local rings at all points y' in its special fiber over X' , or equivalently the *completion* of each such local ring is regular.

Letting x' and x be the respective closed points of the local X' and X , the artinian quotients $\mathcal{O}_{Y',y'}/\mathfrak{m}_{y'}^{n+1}$ of those local rings are captured by infinitesimal special fibers $Y'_n = Y \bmod \mathfrak{m}_{x'}^{n+1}$ of $Y' \rightarrow X'$, and $Y'_n = Y_n := Y \bmod \mathfrak{m}_x^{n+1}$ because $Y' = Y \otimes_A \hat{A}$ with $\hat{A}$ and A having the *same* artinian quotients. In other words, $Y'_0 \simeq Y_0$ and for each $y' \in Y'_0$ and the corresponding $y \in Y_0$ the natural map $\mathcal{O}_{Y,y}^\wedge \rightarrow \mathcal{O}_{Y',y'}^\wedge$ is an isomorphism. But the completed local rings on Y are regular since Y is regular, so we are done! $\square$

Lecture 5

Preliminary reduction steps

§5.1 A note on modifications

Before getting to the reduction steps it will be useful to recall one final general bit of terminology that will come up in our discussion of deJong's proof.

DEFINITION 5.1.1. A **modification** of an integral noetherian scheme S is a proper birational map $\psi : S' \rightarrow S$ with S' integral.

For a finite-type separated S -scheme X that is integral and a modification $\psi : S' \rightarrow S$ with respective generic points $\eta' \mapsto \eta$ (so $\kappa(\eta') = \kappa(\eta)$), via the fiber product diagram

$$\begin{array}{ccc} X \times_S S' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

the **strict transform** X' of $X \rightarrow S$ with respect to ψ is the schematic closure of $X_\eta \otimes_{\eta} \eta'$ (the generic fiber) in $X \times_S S'$.

This notion of “strict transform” is only of interest when $X \rightarrow S$ is surjective (it is empty otherwise). The reason for such terminology is out of analogy with the role of schematic closure for describing blow-ups classically: if $U \subset S$ is the dense open over which ψ is an isomorphism then the strict transform is the schematic closure in $X \times_S S'$ of $X_U = X_U \times_U \psi^{-1}(U)$, and classically the blow-up of a closed subscheme $X \subset \mathbf{P}_k^n$ at a point $p \in X$ was defined to be the schematic closure of $X - \{p\}$ in $\text{Bl}_p(\mathbf{P}_k^n)$ (which sits isomorphically over $\mathbf{P}_k^n - \{p\}$). That this latter description agrees with the modern notion of blow-up is discussed in Appendix A (see in particular Lemma A.3.3 and Example A.2.5).

EXERCISE 5.1.A. When the structure morphism $X \rightarrow S$ is surjective, show that the strict transform X' must be integral, and moreover the natural map $X' \rightarrow X$ is a modification.

REMARK 5.1.2. If $\mathcal{X} \subset X \times_S S'$ is a closed S' -flat subscheme with full generic fiber then $\mathcal{X}$ is the strict transform of X since any S' -flat scheme is the schematic closure of its own generic fiber (as we see by calculation with integral affine opens). This will be useful later.

§5.2 Reduction steps

Our goal here is to reduce to the case of X being a normal projective variety of dimension > 1 over an algebraically closed field. Here we will recall the statement of Theorem 1.2.4 for convenience.

THEOREM (4.1 in [deJ96]). For k any field, let X be a k -variety (integral separated of finite type) and pick any proper closed subset $Z \subset X$. Then we have a diagram

$$\begin{array}{ccccc} \varphi_1^{-1}(Z) & \hookrightarrow & X_1 & \xhookrightarrow{j} & \overline{X}_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_1 & & \\ Z & \hookrightarrow & X & & \end{array}$$

where $\varphi_1 : X_1 \rightarrow X$ is an alteration, $\overline{X}_1$ is a regular projective k -variety and $j : X_1 \hookrightarrow \overline{X}_1$ is a dense open immersion such that the closed set $\varphi_1^{-1}(Z) \cup (\overline{X}_1 - X_1) = \overline{X}_1 - \varphi_1^{-1}(X - Z)$ with the reduced structure is an sncd in $\overline{X}_1$.

REMARK 5.2.1. It will turn out that X_1 is going to be in fact generically smooth over some finite extension K/k . Even if k is algebraically closed in $k(X)$, we *cannot* rule out $K \neq k$; i.e X_1 may not be geometrically integral over k .

Our strategy will be to add constraints step by step, each one building on the last. A recurring motif will be that if we can "modify" X (in the sense of Definition 5.1.1) to add some desired constraint, then the properties of the theorem holding in the modification will descend down to X , thus allowing us to WLOG assume the constraint holds.

We are first going to proceed by induction on $d = \dim(X)$.

Reducing to the case $d := \dim(X) > 0$

If X is 0-dimensional then it has to be a single reduced point $\text{Spec}(K)$ for some finite K/k , and Z is empty (since it was assumed to be a *proper* closed subset), so $\overline{X}_1 = X_1 = X$ suffices. Now we may and do assume $d \geq 1$.

Reducing to the case $k = \overline{k}$

Let us grant the full result in dimension $d \geq 1$ over $\overline{k}$, and we want to deduce it over k in dimension d . Let us consider $(X_{\overline{k}})_{\text{red}}$, which is of pure-dimension d , since X is irreducible (of finite type over k) and was assumed to be of dimension d . Pick $X' \subset (X_{\overline{k}})_{\text{red}}$ an irreducible component given the reduced structure, so it is a $\overline{k}$ -variety of dimension d , and we know the natural map $\pi : X' \rightarrow X$ is a surjection (since $X_{\overline{k}} \rightarrow X$ is flat and thus carries all generic points onto the unique generic point of X), so in particular we must have $Z' := \pi^{-1}(Z) \subsetneq X'$. Applying deJong's theorem to $Z' \subsetneq X'$ gives

$$\begin{array}{ccc} X'_1 & \xhookrightarrow{j_1} & \overline{X}'_1 \\ \downarrow \varphi_1 & & \\ X' & & \end{array}$$

with φ_1 a generically étale alteration (as $\bar{k}$ is perfect), $\bar{X}'_1$ smooth and projective over $\bar{k}$ (since smoothness and regularity are equivalent over algebraically closed fields!), and for $Z'_1 := \varphi_1^{-1}(Z')$ the closed set $Z'_1 \cup (\bar{X}'_1 - X'_1)$ is an sncd in $\bar{X}'_1$.

We want to descend all of this data down to some finite extension of k . To do this we make use of the thorough exposition in [EGA, IV₃, §8, 9, 11, etc.] on “direct limit formalism”. Roughly speaking, this says that any “finitely presented algebro-geometric data” over a ring of the form $A = \varinjlim_{\alpha} A_{\alpha}$ descends to some A_{α_0} , along with “all the properties” (at the cost of making α_0 a bit bigger). This is made precise in [EGA], and it provides a finite extension K/k such that:

- (1) $(X_{\bar{k}})_{\text{red}} = ((X_K)_{\text{red}}) \otimes_K \bar{k}$.
- (2) There exists an irreducible component $X'' \subset (X_K)_{\text{red}}$ such that $X'' \otimes_K \bar{k} = X'$ and $Z'' \subsetneq X''$ (the preimage of Z) satisfies $Z'' \otimes_K \bar{k} = Z'$ topologically (inside $X_{\bar{k}}$).
- (3) We have a diagram of K -schemes

$$\begin{array}{ccc} X''_1 & \xrightarrow{j_1''} & \bar{X}''_1 \\ \downarrow \varphi_1''' & & \\ X'' & & \end{array}$$

where φ_1'' is a generically étale alteration and $\bar{X}''_1$ is a K -smooth (hence regular!) projective variety over K , with $Z''_1 \cup (\bar{X}''_1 - X''_1)$ an sncd in $\bar{X}''_1$ for $Z''_1 := (\varphi_1'')^{-1}(Z)$ (in fact, this sncd closed subset is even a union of K -smooth closed subvarieties of codimension 1 with all intersections among them K -smooth of the “expected” codimension; we then relax K -smoothness to regularity, as is all one needs for being an sncd).

Since the finite surjection $(X_K)_{\text{red}} \rightarrow X$ is finite flat over a dense open in the integral X and $X'' \subset (X_K)_{\text{red}}$ is an irreducible component, the composite map $X'' \rightarrow X$ is finite and generically flat (recall every generic point of X_K must sit over the unique generic point in X !). Thus the composite map

$$X''_1 \xrightarrow{\varphi_1''} X'' \rightarrow X$$

is an alteration that gives us what we wanted, since $\bar{X}''_1$ being projective over K implies it is projective over k as well.

This completes the reduction to the case $k = \bar{k}$.

Reducing to the case of X projective

By the version of Chow’s Lemma in [EGA, II, §5.6] (for separated finite-type schemes over a noetherian base; i.e., not requiring a properness hypothesis) there exists a quasi-projective variety X' over k with a modification $\pi : X' \rightarrow X$. We may replace $Z \subsetneq X$ with $\pi^{-1}(Z) \subsetneq X'$ to thereby reduce to the quasi-projective case.

Embed a quasi-projective X via an open immersion into a projective closure $\overline{X}$, in which Z has closure denoted $\overline{Z}$ (this meets the open subscheme $X \subset \overline{X}$ in Z). We may replace (X, Z) with $(\overline{X}, \overline{Z} \cup (\overline{X} - X))$ to pass to the projective case because if we have a generically étale alteration $\overline{\varphi} : \overline{X}' \rightarrow \overline{X}$ for a smooth projective $\overline{X}'$ satisfying the desired properties (including relative to $\overline{Z}$) then via the commutative diagram

$$\begin{array}{ccc} X' & \hookrightarrow & \overline{X}' \\ \downarrow & & \downarrow \overline{\varphi} \\ X & \hookrightarrow & \overline{X}, \end{array}$$

(with $X' := \overline{\varphi}^{-1}(X)$) the restriction $\varphi : X' \rightarrow X$ and open immersion $j : X' \hookrightarrow \overline{X}'$ do the job for X .

Reducing to the case Z is the support of an effective Cartier divisor

Looking at Z as a reduced closed subscheme of X , we can blow up X along the ideal sheaf $\mathcal{I}_Z$ to get a morphism

$$\pi : X' := \mathrm{Bl}_Z(X) \rightarrow X.$$

By the standard properties of a blowup, we know that π is a projective modification and (by definition, essentially) the preimage $Z' := \pi^{-1}(Z) \subset X'$ is an effective Cartier divisor. Moreover $Z' \subsetneq X'$, since π is surjective (as it is proper and an isomorphism over the open $X - Z$ that is dense in the integral X). Thus we may replace (X, Z) with (X', Z') .

For any modification $\pi : X' \rightarrow X$ with X' a projective variety, the preimage Z' of Z in X' is also the support of an effective Cartier divisor, since for non-empty affine open $\mathrm{Spec}(A) \subset X$ and non-empty affine open $\mathrm{Spec}(B) \subset \pi^{-1}(\mathrm{Spec}(A))$ any nonzero $f \in A$ has image in B that is nonzero (hence not a zero-divisor) since $A \rightarrow B$ is injective (as π is dominant). Thus, given any such π we can always pass to (X', Z') . The final reduction step uses this (as will others):

Reducing to the case X is normal

The normalization map $\widetilde{X} \rightarrow X$ is finite – hence proper – by [Mat90, §33] and it is birational since X is a variety. Thus, this is a modification, so we may replace X with $\widetilde{X}$ and replace Z with its preimage in $\widetilde{X}$.

The case $d = 1$ is now settled because the normal projective variety X is then Dedekind and thus its local rings are dvr's, so Z_{red} is a finite set of closed points each of which is regular and Cartier (as the local rings on X are dvr's, so their maximal ideals are principal).

Lecture 6

Curve Fibration I

§6.1 Controlling the finite-fiber locus and enlarging Z

We have reduced the task to constructing a generically étale alteration $\varphi : X' \rightarrow X$ of a normal projective variety X of dimension $d \geq 2$ over an algebraically closed field k . Although it is generally hopeless to construct φ to be étale or even flat over a specific point $x \in X(k)$, we can at least exert some mild control over the locus of $x \in X$ with finite fibers: its complement has codimension ≥ 2 . This is the content of the following result.

PROPOSITION 6.1.1. Any alteration $\varphi : X' \rightarrow X$ between projective varieties over $k = \bar{k}$ with normal X is finite over a dense open $U \subset X$ with complement of codimension ≥ 2 .

Proof. Let $R = \mathcal{O}_{X,x}$ for a point $x \in X$ with codimension 1; i.e., $\dim R = 1$. By normality, any such R is a discrete valuation ring. We will show that the proper map $X'_R = X \times_X \operatorname{Spec}(R) \rightarrow \operatorname{Spec}(R)$ obtained by localization to R is a finite morphism. Once this is shown, by “spreading out” principles we obtain an open neighborhood $V \subset X$ around x such that $\varphi^{-1}(V) \rightarrow V$ is finite. The non-empty union U of all such V ’s is then an open subset of X for which $\varphi^{-1}(U) \rightarrow U$ is finite (as may be verified over each member V of an open cover of U), and the proper closed set $X - U$ contains no points in X with a 1-dimensional local ring on X , so $X - U$ has codimension ≥ 2 at all of its points.

Noting that $X'_R \rightarrow \operatorname{Spec}(R)$ has generic fiber $\eta' \rightarrow \eta$ that is finite, we are reduced to proving finiteness of any proper map $f : Y \rightarrow \operatorname{Spec}(R)$ between integral schemes for which R is a discrete valuation ring with module-finite integral closure in all finite – not necessarily separable! – extensions of its fraction field (e.g., $\mathcal{O}_{X,x}$ as above, by spreading-out from module-finiteness for integral closures with affine varieties over a field [Mat90, §33]) and the generic fiber Y_η is η -finite. Finiteness for f is the same as quasi-finiteness since f is proper, so our task concerns only the closed fiber Y_0 of Y : we just need to check that it has only finitely many points. Let R' be the R -finite normalization of R in the finite extension $k(\eta')$ of the fraction field $k(\eta)$ of R . In particular, R' is a semi-local Dedekind domain.

By the valuative criterion for properness, adapted to the case of Dedekind domains (rather than just discrete valuation rings), the given map $\eta' \rightarrow Y$ over $\operatorname{Spec}(R)$ extends uniquely to an R -map $h : \operatorname{Spec}(R') \rightarrow Y$. This latter map between integral schemes is an isomorphism between generic fibers over R , so it is dominant, and it is also proper because it is a map between proper R -schemes. But a dominant proper map is surjective, so h is surjective. Hence, Y_0 is the image of the special fiber of $\operatorname{Spec}(R')$ over $\operatorname{Spec}(R)$. The R -finiteness of R' then ensures that Y_0 consists of only finitely many points, as desired. $\square$

REMARK 6.1.2. As a final general-nonsense observation before we take up the geometric arguments of deJong, we discuss why enlarging Z is harmless. Suppose that $Z \subset \tilde{Z} \subsetneq X$ is a containment of proper closed subsets with Z the support of a Cartier divisor in X and there exists a regular alteration $\varphi : X' \rightarrow X$ such that $\varphi^{-1}(\tilde{Z})_{\text{red}} \subset X'$ is an sncd. Since $\varphi^{-1}(\tilde{Z})_{\text{red}}$ is then a union of regular closed subschemes with pure codimension 1 whose successive intersections are regular with the “expected” dimension, any union among its reduced irreducible components is also of that type and hence is also an sncd. Thus, for dimension reasons we get $\varphi^{-1}(Z)_{\text{red}}$ is an sncd. This freedom to increase Z in the argument (to replace it with a larger Cartier divisor, for example) will be used a lot later.

§6.2 A model case of curve fibration via blow-up

We now give a “warm-up” example of fibering a variety in curves after a blow-up, using $X = \mathbf{P}^d$. Pick $p \in \mathbf{P}^d(k)$. Taking a hyperplane $H = \mathbf{P}^{d-1}$ in $\mathbf{P}^d$ not containing p , each line ℓ in $\mathbf{P}^d$ passing through p corresponds to exactly one point in H (namely the point $\ell \cap H$). This makes H represent the functor of “lines in $\mathbf{P}^d$ passing through p ”, the rigorous details of which we leave to the reader. Thinking about the case $p = [1, 0, \dots, 0]$ and $H = \{x_0 = 0\}$ identifies this scheme of lines through p with $\mathbf{P}^{d-1}$.

There is also the classical description

$$\text{Bl}_p(\mathbf{P}^d) = \{(q, \ell) \in \mathbf{P}^d \times \mathbf{P}^{d-1} \mid q \in \ell\}$$

(as a closed subscheme of $\mathbf{P}^d \times \mathbf{P}^{d-1}$) via a *unique* isomorphism over $\mathbf{P}^d$, identifying the blow-up map with the first projection $\varphi : \text{Bl}_p(\mathbf{P}^d) \rightarrow \mathbf{P}^d$ that is an isomorphism over $\mathbf{P}^d - \{p\}$.

The second projection $\pi : \text{Bl}_p(\mathbf{P}^d) \rightarrow \mathbf{P}^{d-1}$ is a Zariski $\mathbf{P}^1$ -bundle, as one can check over open affine spaces (again after passing to the concrete case $p = [1, 0, \dots, 0]$ and $H = \{x_0 = 0\}$), most concretely described pointwise by the observation

$$\pi^{-1}(\{\ell\}) = \ell \times \{\ell\} \subset \mathbf{P}^d \times \{\ell\}$$

as a scheme-theoretic fiber (check!).

Now consider a reduced (possibly reducible) proper closed $Z \subsetneq \mathbf{P}^d$, and pick $p \notin Z$, so φ restricts to an isomorphism $\varphi^{-1}(Z) \xrightarrow{\sim} Z$. Let π_Z be the restriction of π to $\varphi^{-1}(Z)$ (which is henceforth identified with Z). Note

$$\pi_Z^{-1}(\{\ell\}) = (\ell \cap Z) \times \{\ell\}$$

as schemes, and this is k -finite. since $p \in \ell$ but $p \notin Z$. This gives π_Z is proper and quasi-finite, hence *finite*.

We expect that if p has been chosen at random then for “most” lines ℓ in $\mathbf{P}^d$ containing p , the

finite scheme $\ell \cap Z$ should be geometrically reduced (and thus étale, as it is k -finite). To show this, we will use the Bertini theorems, to be discussed in some detail next time (referring to [Jou83] for proofs). More broadly:

HOPE. For a “random” choice of p , the finite $\pi_Z : Z \rightarrow \pi(Z)$ is generically étale, and even birational if $\dim Z < d - 1$.

The above ideas will be upgraded to fiber a rather general X of dimension d in curves over $\mathbf{P}^{d-1}$:

LEMMA 6.2.1 (4.11 in [deJ96]). Consider X a projective variety of dimension $d \geq 2$ over an algebraically closed field k and $Z \subsetneq X$ the support of a Cartier divisor. There exists a finite subset $S \subset X^{\text{sm}}(k)$ outside Z and $f : X' := \text{Bl}_S(X) \rightarrow \mathbf{P}_k^{d-1}$ over k such that:

1. all fibers of f are pure dimension 1 and the open relative smooth locus $\text{sm}(X'/\mathbf{P}^{d-1}) \subset X'$ of f is fiberwise dense,
2. for the structure map $\varphi : X' \rightarrow X$ that is an isomorphism over $X - S \supset Z$, the map $Z \xleftarrow{\sim} \varphi^{-1}(Z) \xrightarrow{f} \mathbf{P}_k^{d-1}$ is generically étale and finite.

Also, if X is normal then we can arrange there to exist a dense open $U \subset \mathbf{P}_k^{d-1}$ such that $f^{-1}(U) \rightarrow U$ has smooth geometrically connected fibers.

To prove this we will find a suitable finite and generically étale $X \xrightarrow{h} \mathbf{P}^d$ and take $S = h^{-1}(p)$ for a “good” $p \in \mathbf{P}_k^d$ well-positioned outside $h(Z) \subset \mathbf{P}_k^d$.

For now we just sketch the idea of the proof, with details to be discussed next time. We shall use a Projective Noether Normalization expressing X as finite over a projective d -space. The finite map will be built as a composition of successive projections away from points into successive hyperplanes. To get started, since X is projective we can pick a closed immersion of it into $\mathbf{P}_k^N$ for some N . If $N = d$ then $X = \mathbf{P}_k^d$ and so we can use the “warm-up” example.

Suppose $N > d$. Pick a k -point $p \in \mathbf{P}_k^N - X$. As in the “warm-up” example, identify $\mathbf{P}_k^{N-1}$ as the scheme of lines in $\mathbf{P}_k^N$ passing through p . Consider the k -map $\pi_p : \mathbf{P}_k^N - \{p\} \rightarrow \mathbf{P}_k^{N-1}$ taking q to the unique line through q and p (this is readily defined rigorously as an algebraic map, as usual by passing to the case $p = [1, 0, \dots, 0]$ and the hyperplane $\{x_0 = 0\}$ avoiding p). Let $\pi_{p,X}$ be the restriction of π_p to $X \subset \mathbf{P}_k^N - \{p\}$. Note $\pi_p^{-1}(\{\ell\}) = \ell - \{p\}$ *as schemes* (i.e., using scheme-theoretic fiber structure), so $\pi_{p,X}^{-1}(\{\ell\}) = (\ell - \{p\}) \cap X = \ell \cap X$ *as schemes* too.

Note that $\pi_{p,X} : X \rightarrow \mathbf{P}_k^{N-1} = \mathbf{P}^d$ is quasi-finite (as $\ell \cap X \subsetneq \ell$ since $p \notin X$) and proper, hence finite. If $d = N - 1$ and (as we hope to happen for a random p relative to X) “most” lines ℓ containing p have $\ell \cap X$ geometrically reduced (and hence étale since it is of dimension 0), the finite map $\pi_{p,X}$ is generically étale onto $\mathbf{P}_k^{N-1}$. If instead $d \leq N - 2$ then we hope for “most” p that $\pi_{p,X} : X \rightarrow \pi(X) \subset \mathbf{P}_k^{N-1}$ is birational onto its image, as is $Z \rightarrow \pi(Z)$ (beware Z may be reducible, so birationality requires being attentive to different irreducible components not landing

on top of each other under π_p). When this is the case then by using such a p we can reduce to $\pi(Z) \subset \pi(X) \subset \mathbf{P}_k^{N-1}$ and proceed by downward induction on N .

Lecture 7

Curve Fibrations II

First we give a coordinate free version of the projection $\pi_p : \mathbf{P}^N - \{p\} \rightarrow H \simeq \mathbf{P}^{N-1}$ to the space of lines through p . Let V be an $(N+1)$ -dimensional k -vector space and $W \subset V$ a hyperplane, so V/W is 1-dimensional. Note $\mathbf{P}(V) = \text{Proj}(\text{Sym}(V))$ is the scheme representing the functor $S \mapsto \{V_S \twoheadrightarrow \mathcal{L}\} / \simeq$ where $\mathcal{L}$ runs over invertible $\mathcal{O}_S$ -modules. As for any proper subspace $W \subset V$ we have a natural closed immersion $\mathbf{P}(V/W) \hookrightarrow \mathbf{P}(V)$ and a point $p \in \mathbf{P}(V/W)(k)$ is represented by a nonzero $\lambda_0 : V \rightarrow k$ with $\lambda_0(W) = 0$. Thus, we get a well-defined morphism $\pi_W : \mathbf{P}(V) - \mathbf{P}(V/W) \rightarrow \mathbf{P}(W)$ defined functorially by $\lambda \mapsto \lambda|_W$ (since $V_S \twoheadrightarrow \mathcal{L}$ being nonzero on W_s for all $s \in S$ expresses that the corresponding S -point in $\mathbf{P}(V)$ is disjoint from $\mathbf{P}(V/W)$). For W a hyperplane, this is exactly π_p for the unique $p \in \mathbf{P}(V/W)(k)$.

§7.1 Projection away from a well-chosen point

To prove Lemma 6.2.1, we will express X finitely over a projective d -space via a composition of $N-d$ well-chosen projections in succession. This is based on successive applications of the following result whose proof occupies the rest of this lecture and next time:

PROPOSITION 7.1.1 (§2.11 of [deJ96]). Let k be a field, $X \subset \mathbf{P}_k^N$ a closed generically smooth subscheme of pure dimension $d < N$. There exists a dense open $U \subset \mathbf{P}_k^N - X$ such that for all finite separable k'/k and $p \in U(k')$ (so $p \notin X_{k'}$) the finite map $\pi_p : X_{k'} \rightarrow \mathbf{P}_k^{N-1}$ satisfies

- (α) π_p is birational onto its image if $d \leq N-2$,
- (β) π_p is generically étale onto $\mathbf{P}_k^{N-1}$ if $d = N-1$.

Grant there exists such a U' for $X_{k_s} \subset \mathbf{P}_{k_s}^N$; we will build our desired U over k . By chasing coefficients in the (finite) set of homogeneous polynomials defining the closed complement of U' in $\mathbf{P}_{k_s}^N$, there exists a finite Galois K/k inside k_s and open $U \subset \mathbf{P}_K^N$ such that $U \otimes_K k_s = U'$. Then U works as our desired U over K .

Consider $\bigcap_{\gamma \in \text{Gal}(K/k)} \gamma^*(U) \subset \mathbf{P}_K^N$. This is $\text{Gal}(K/k)$ -stable, nonempty, and open. Thus, by Galois descent $\mathcal{U} = U_K$ for some open $U \subset \mathbf{P}_k^N$, which is readily checked to do the job of our desired U . Therefore, we can assume that $k = k_s$, so now any smooth k -scheme Y has $Y(k)$ Zariski-dense in Y .

REMARK 7.1.2. For (β), it suffices to check on a dense open subset of the dense open $X^{\text{sm}} \subset X$, so it is equivalent to treat all irreducible components X_i of X separately (as we can then pass to $\cap_i U_i$ as U). The assertion in (α), however, is too global to permit such a reduction to

irreducible components since birationality requires pairwise distinctness for the closed images $\pi_p(X_i)$.

We first present the idea of a proof of 7.1.1 for $d = N - 1$. Let $V = X^{\text{sm}}$ with irreducible components X_i , so for a line $\ell \subset \mathbf{P}^N$ through $p \notin X$, if ℓ misses $X - X^{\text{sm}}$ (as we expect for typical ℓ since $X - X^{\text{sm}}$ has codimension at least 2 in $\mathbf{P}^N$) then $\ell \cap X = \ell \cap V$ as schemes. We would like $\ell \cap V$ to be étale and meet *every* V_i for “most” ℓ through “most” $p \in X$. To do this we need the Bertini theorems (for which [Jou83] is an excellent reference, as we’ll discuss later).

§7.2 Bertini theorems and an application

Here is the setup for Bertini theorems. Let F be a field and $Z \subset \mathbf{P}_F^N$ a locally closed subscheme of pure dimension $d \geq 1$. Let $\mathbf{G} = \text{Gr}(r, N)$ be the Grassmanian of codimension- r subspaces of $\mathbf{P}_F^N$ for a fixed $1 \leq r \leq d$ (e.g. if $d = N - 1$ and $r = N - 1$, then $\mathbf{G}$ is the Grassmanian of lines in $\mathbf{P}^N$: all lines, not just those through a fixed F -point). Let

$$\mathcal{V} = \{(p, V) \in \mathbf{P}^N \times \mathbf{G} \mid p \in V\} \subset \mathbf{P}_F^N \times \mathbf{G}$$

be the “universal codimension- r linear subspace” of $\mathbf{P}^N$ over F . More precisely, over $\mathbf{G}$ we have the universal quotient vector bundle $\mathcal{O}_{\mathbf{G}}^{N+1} \rightarrow \mathcal{E}$ of rank $r + 1$ and $\mathcal{V} = \mathbf{P}(\mathcal{E}) \subset \mathbf{P}(\mathcal{O}_{\mathbf{G}}^{N+1}) = \mathbf{P}^{N+1} \times \mathbf{G}$ a Zariski $\mathbf{P}^r$ -bundle over $\mathbf{G}$ linearly inside $\mathbf{P}^N \times \mathbf{G}$. Consider

$$\begin{array}{ccc} \mathcal{V} \cap Z_{\mathbf{G}} & \hookrightarrow & Z_{\mathbf{G}} \\ & \searrow & \swarrow \\ & \mathbf{G} & \end{array} \quad (\dagger)$$

For a field F'/F and $L \in \mathbf{G}(F')$ a codimension- r linear subspace of $\mathbf{P}_{F'}^N$, the diagram $\dagger$ pulls back to

$$\begin{array}{ccc} L \cap Z_{F'} & \hookrightarrow & Z_{F'} \\ & \searrow & \swarrow \\ & \text{Spec}(F') & \end{array}$$

Informally, as we vary through all field-valued points of $\mathbf{G}$ over F , the fibers of $\mathcal{V} \cap Z_{\mathbf{G}}$ vary through the slices of Z by codimension- r linear subspaces of $\mathbf{P}^N$. The idea of “Bertini theorems” is that the locus in $\mathbf{G}$ over which such slices inherit various reasonable “geometric” properties of Z should contain a dense open subset of $\mathbf{G}$. This is made precise for some key properties as follows (with references given in [Jou83] for proofs):

THEOREM 7.2.1 (Bertini Theorems). 1. [Jou83, Thm. 6.10(2), Cor. 6.11(1b)] If Z is F -smooth there exists a dense open $\Omega_1 \subset \mathbf{G}$ such that $Z_{F'} \cap L$ is non-empty and smooth of pure

dimension $d - r$ for all F'/F and $L \in \Omega_1(F')$.

2. [Jou83, Thm. 6.10(2)] If Z is geometrically reduced over F then there exists a dense open $\Omega_2 \subset \mathbf{G}$ such that $Z_{F'} \cap L$ is non-empty and geometrically reduced of pure dimension $d - r$ for all F'/F and $L \in \Omega_2(F')$.
3. [Jou83, Thm. 6.10(3)] If Z is geometrically irreducible and $r \leq d - 1$ then there exists a dense open $\Omega_3 \subset \mathbf{G}$ such that $Z_{F'} \cap L$ is geometrically irreducible of pure dimension $d - r$ for all F'/F and $L \in \Omega_3(F')$.

REMARK 7.2.2. (i) [Jou83, Thm. 6.10(1)] If $r > d$ then there exists a dense open $\Omega \subset \mathbf{G}$ such that $Z_{F'} \cap L = \emptyset$ for all F'/F and $L \in \Omega(F')$.

- (ii) The proof of (3) is global (in the sense that geometric irreducibility is not a property of local rings), unlike that of (1) and (2).

Let us return to Proposition 7.1.1 over $F = k = k_s$ and $d = N - 1$. Apply Theorem 7.2.1(2) to the connected (= irreducible) components of X^{sm} with $r = d - 1$ and apply Remark 7.2.2(i) to $X - X^{\text{sm}}$: for “most” lines (i.e. all lines in a dense open $\Omega \subset \mathbf{G}$) the scheme-theoretic intersection $\ell \in \mathbf{P}_k^N$, $\ell \cap X = \ell \cap X^{\text{sm}}$ is geometrically reduced, hence étale (as the intersection is of pure dimension 0), and meets every connected component of X^{sm} (as all are of pure codimension 1).

Pick $\ell_0 \in \Omega(k)$, and a k -point $p \in \ell_0 - (X \cap \ell_0)$ (as we may find since k is infinite). For $\pi_p : X \rightarrow \mathbf{P}_k^{N-1}$ we have the scheme-theoretic fiber $\pi_p^{-1}(\{\ell_0\}) = \ell_0 \cap X$ that is étale and meets every irreducible component of X^{sm} !

Pick x_i in the fiber over $\{\ell_0\}$ belonging to $X^{\text{sm}} \cap X_i$. Since $\ell_0 \cap X = \ell_0 \cap X^{\text{sm}}$ is étale over k , the fiber $\Omega_{X/\mathbf{P}^{N-1}}^1(x_i)$ of the coherent sheaf $\Omega_{X/\mathbf{P}^{N-1}}^1$ can be computed by first pulling back along the k -point of $\mathbf{P}^{N-1}$ corresponding to ℓ_0 : it is $\Omega_{(\ell_0 \cap X)/k}^1(x_i)$. But this vanishes since $\ell_0 \cap X$ is étale! Hence, the coherent $\Omega_{X/\mathbf{P}^{N-1}}^1$ has vanishing fiber at x_i , so by Nakayama’s Lemma it has vanishing stalk at x_i and thus vanishes on an open neighborhood of x_i in X .

Varying over all i , we get a dense open in X on which $\Omega_{X/\mathbf{P}^{N-1}}^1$ vanishes. But the finite map $X \rightarrow \mathbf{P}^{N-1}$ is flat on X^{sm} by Miracle Flatness (recall X has pure dimension $d = N - 1$), so π_p is étale at the x_i ’s (since in the presence of flatness, étaleness can be checked on the fibers). Thus, by openness of the étale locus we have established the generic étaleness of π_p .

We have constructed *some* p as desired in Proposition 7.1.1 for $d = N - 1$. It feels intuitively clear that the p ’s obtained in this way should sweep out at least some non-empty open in $\mathbf{P}_k^N - X$. Next time we’ll review the standard technique with “incidence correspondences” to justify such intuition rigorously, and then we’ll move on to proving Proposition 7.1.1 when $d \leq N - 2$.

Lecture 8

Bertini and Birational Projections

§8.1 Incidence correspondence: wrap-up for $d = N - 1$

In the last lecture, for $X \subset \mathbf{P}_k^N$ closed of pure dimension $d = N - 1$ with $k = k_s$ we gave a dense open $\Omega \subset \mathbf{G} := \text{Gr}(N - 1, N)$ such that for $\ell_0 \in \Omega(k)$ and a k -point $p \in \ell_0 - (\ell_0 \cap X)$, the finite projection $\pi_p : X \rightarrow \mathbf{P}^{N-1}$ is étale on a dense open. To complete the proof of Proposition 7.1.1 for $d = N - 1$ we have to show that the p 's with this latter property sweep out at least the k -points of dense open in $\mathbf{P}_k^N - X$. There really is something to prove here (i.e., it is not a tautology, though we aren't claiming it is rocket science either) since the way we made p is by first picking a “generic” ℓ_0 in $\mathbf{G}$ and then picking $p \in \ell_0 - (\ell_0 \cap X)$ depending on that choice. To show that such p 's exhaust at least the k -points of a dense open set in $\mathbf{P}_k^N - X$, we are going to use the very handy notion of *incidence correspondence*.

Consider the “universal line”

$$L := \{(p, \ell) \in \mathbf{P}^N \times \mathbf{G} \mid p \in \ell\} \subset \mathbf{P}^N \times \mathbf{G}.$$

This is called an “incidence correspondence” since it encodes (scheme-theoretically: we leave it to the reader to express “ $p \in \ell$ ” in scheme-theoretic terms via the functorial description of $\mathbf{G}$) a containment relation among geometric objects inside $\mathbf{P}^N$. The second projection $\text{pr}_2 : L \subset \mathbf{P}^N \times \mathbf{G} \rightarrow \mathbf{G}$ makes L into a Zariski $\mathbf{P}^1$ -bundle, so L is a variety (as $\mathbf{G}$ is a variety), as we leave to the reader to check. For intuition, observe that at the level of field-valued points we have $\text{pr}_2^{-1}(\{\ell\}) = \ell \subset \mathbf{P}_k^N$ for each k -point $\{\ell\} \in \mathbf{G}(k)$.

The first projection $\text{pr}_1 : L \subset \mathbf{P}^N \times \mathbf{G} \rightarrow \mathbf{P}^N$ makes L into a Zariski $\mathbf{P}^{N-1}$ -bundle, again as we leave it to the reader to check. Once again for intuition, consider field-valued points: the pr_1 -fiber over $p \in \mathbf{P}^N(k)$ is the space of lines in $\mathbf{P}^N$ through p , which we know is a copy of $\mathbf{P}_k^{N-1}$. As a projective-space bundle map, pr_1 is certain open map and so $\text{pr}_1(\text{pr}_2^{-1}(\Omega))$ is open. But the overlap of this with $\mathbf{P}^N - X$ is *exactly* the collection of p 's obtained by our argument near the end of last time (at the level of k -points), so we get the desired openness and hence are done when $d = N - 1$.

§8.2 Application of higher-dimensional Bertini

Now consider $d \leq N - 2$. We will use a higher-dimensional Bertini theorem: for “good” $W \subset \mathbf{P}^N$ of codimension d (so dimension $N - d \geq 2$) such that $W \cap X$ is “nice” we will then search among lines in W . Let V_i be the irreducible (= connected) components of X^{sm} , which is of pure dimension d . We have $V_i = (X_i)^{\text{sm}} - \bigcup_{j \neq i} X_i \cap X_j$ for X_i the irreducible components of X . For $\mathbf{G} := \text{Gr}(d, N)$ the Bertini theorems from last time provide many $W \in \mathbf{G}(k)$ such that $W \cap X = W \cap X^{\text{sm}}$ (this

says W is disjoint from $X - X^{\text{sm}}$ which has dimension $< d$) and each $W \cap X_i = W \cap V_i$ is non-empty and étale. Since $k = k_s$, each $W \cap V_i$ is a finite non-empty collection of k -points. Pick $q_j \in W \cap V_j$ and a line $\ell_j \subset W$ through q_j that misses all other finitely many points in $W \cap X$ (such a line can be found since W is a projective space with dimension $N - d \geq 2$ and k is infinite).

Since $\ell_j \subset W$, we have $\ell_j \cap X = \ell_j(W \cap X) = \{q_j\}$ as schemes. Thus, for $p_j \in \ell_j - (\ell_j \cap X) = \ell_j - \{q_j\}$ the *finite* (not at all generically flat, since $d < N - 1$) projection $\pi_{p_j} : X_j \rightarrow \mathbf{P}_k^{N-1}$ to the space of lines through q_j has fiber over $\{\ell_j\}$ equal to $\ell_j \cap X_j = \{q_j\}$. That is, π_{p_j} has fiber-degree 1 over a k -point. This will yield birationality of X_j onto its image under π_{p_j} because for a *finite* surjective map (such as $X_j \rightarrow \pi_{p_j}(X_j)$) it is a fact that the fiber-degree can only go *up* under specialization (and hence having fiber-degree 1 over some k -point forces fiber-degree 1 over the generic point of the image; i.e., degree 1 for the extension of function fields of the varieties X_j and $\pi_{p_j}(X_j)$):

LEMMA 8.2.1. If $f : X \rightarrow Y$ is a finite surjective map of schemes and $y, \eta \in Y$ are points with $y \in \overline{\{\eta\}}$ then $\deg_{\kappa(y)}(X_y) \geq \deg_{\kappa(\eta)}(X_\eta)$.

Proof. We give two proofs: a short one via Nakayama's Lemma pointed out by Daniel Kim, and a more involved proof in the noetherian case (all we need) using the powerful trick of “base change to dvr's” that is widely useful (as is Nakayama's Lemma, but in a very different way).

We can localize to y so that $Y = \text{Spec}(A)$ is local and then $X = \text{Spec}(B)$ for an A -algebra B that is module-finite over A . We can also pass to the quotient of A by the ideal corresponding to η , so A is a local domain, say with fraction field K . If k is the residue field of A and $r = \dim_k B/\mathfrak{m}B$ then Nakayama's Lemma provides an A -linear surjection $A^r \twoheadrightarrow B$. Localizing to the fraction field of A then gives that B_K is a quotient of K^r as a K -vector space, so $\dim_K B_K \leq r = \dim_k B_k$ as desired.

For the second proof in the noetherian case, we may assume $y \neq \eta$ and we invoke the very handy fact that there exists a dvr R and a map $\text{Spec}(R) \rightarrow Y$ carrying the closed point to y and the generic point to η . (One way to make this is to blow up the closure of η at y and take a discrete valuation of the function field dominating the 1-dimensional local ring at a generic point of the exceptional divisor; recall from the Krull-Akizuki theorem [Mat90, Thm. 11.7] that the integral closure of a 1-dimensional noetherian domain is always Dedekind even without module-finiteness of the normalization.) By design we can arrange that R has fraction field $\kappa(\eta)$ but its residue field is generally a massive extension of $\kappa(y)$ (typically with positive transcendence degree).

The fiber-degree for a finite map is unaffected by residue field extension, so for the purposes of what we are trying to prove we may base change along $\text{Spec}(R) \rightarrow Y$ (which tends to hugely increase the residue field at the special point) to reduce to the case that $Y = \text{Spec}(R)$ for a dvr R . But then $X = \text{Spec}(B)$ for an R -finite B , so by the structure theorem for finitely generated modules over a dvr we have as R -modules that $B = R^m \oplus T$ for some $m \geq 0$ and a torsion R -module. Then $B_K = K^m$ and $B_k = k^m \oplus (T/\pi T)$ for a uniformizer π of R . Hence, $\dim_k B_k \geq m = \dim_K B_K$ as

desired. (More broadly, when seeking to prove results relating *geometric* properties of X_y and X_η for a finite type map $X \rightarrow Y$ with noetherian Y and $y \in \overline{\{\eta\}}$ then one can use such a base change trick to reduce to the case $Y = \text{Spec}(R)$ for a dvr R .) $\square$

Now we find $p \in (\mathbf{P}^N - X)(k)$ for which the finite $\pi_p : X \rightarrow \mathbf{P}^{N-1}$ is birational onto its image by harnessing the preceding work. Pick $W \in \mathbf{G}(k)$ such that $W \cap X = W \cap X^{\text{sm}}$ with each $W \cap X_i$ non-empty and étale (“most” $W \in \mathbf{G}(k)$ satisfy these requirements). In the linear space $W \subset \mathbf{P}^N$ of dimension $N - d \geq 2$, there are finitely many lines joining the distinct pairs of points in $W \cap X$. Choose $p \in (W - W \cap X)(k)$ not on any of those lines, and for each irreducible component X_i pick $q_i \in (W \cap X_i)(k) \subset X^{\text{sm}}$, so the line $\ell_i := pq_i \subset W$ meets X *only* at q_i (as otherwise p would be on one of the finitely many lines in W joining distinct points in $W \cap X$, contrary to how p was chosen), and scheme-theoretically $\ell_i \cap X = \{q_i\}$. Thus, the finite map $\pi_p : X \rightarrow \mathbf{P}_k^{N-1}$ to the space of lines through p carries X_i onto an image $\pi_p(X_i)$ that contains the k -point $\{\ell_i\}$ since $\pi_p^{-1}(\{\ell_i\}) = \ell_i \cap X = \{q_i\} \in X_i(k)$.

The fact that $\pi_p : X_i \rightarrow \pi_p(X_i)$ has $\{\ell_i\}$ -fiber with k -degree 1 implies birationality. Moreover, since $q_i \notin X_j$ for $j \neq i$ we have $\pi_p(X_i) \neq \pi_p(X_j)$ for $j \neq i$. Thus, $\pi_p : X \rightarrow \pi_p(X)$ is birational too! This completes the construction of a k -point $p \in \mathbf{P}_k^N - X$ as required in Proposition 7.1.1 for $d \leq N - 2$. We leave it to the reader to carry out an incidence correspondence argument (using higher-dimensional Bertini slices) to rigorously justify that the p ’s obtained in this way sweep out at least the k -points in some dense open subset $U \subset \mathbf{P}_k^N - X$. This finishes the proof of Proposition 7.1.1 for $d \leq N - 2$.

Summary. Let’s summarize where we are at toward the proof of Lemma 6.2.1. Consider an algebraically closed field k , a closed subvariety $X \subset \mathbf{P}_k^N$ of dimension $d \geq 2$, and a proper closed subset $Z \subset X$ that is the support of a Cartier divisor. We can use Proposition 7.1.1 for both X and Z (as well as their successive images under ever more projections away from well-chosen points) to build a finite map $\pi : X \rightarrow \mathbf{P}_k^d$ (as a composition of $N - d$ successive projections away from points) that is generically étale and for which $\pi : Z \rightarrow \pi(Z)$ is birational. This is a projective Noether normalization presentation of X that is “well-adapted” to Z .

Next time we shall construct a non-empty finite subset $S \subset X^{\text{sm}} \cap (X - Z)$ that will do the job for Lemma 6.2.1.

Lecture 9

Curve Fibrations III

§9.1 Finding the right blow-up locus

Recalling our work so far, we have a d -dimensional variety X with dimension $d \geq 2$ over an algebraically closed field k and a non-empty proper closed subset $Z \subset X$ that is the support of an effective Cartier divisor in X with the following setup:

$$\begin{array}{ccccc} \mathbf{P}^N & \longleftrightarrow & X & \xrightarrow{\pi} & \mathbf{P}^d \\ & & \uparrow & & \uparrow \\ & & Z & \xrightarrow[\text{birational}]{\pi} & \pi(Z) \end{array}$$

Here π is a finite, generically étale map that we constructed as the composition of $N - d$ “generic” point projections. Let $\Omega \subset \mathbf{P}_k^d$ be a dense open over which $\pi : X \rightarrow \mathbf{P}_k^d$ is étale (so $\pi^{-1}(\Omega)$ inherits smoothness from Ω and so is contained in X^{sm}). Pick $\xi \in \Omega \cap (\mathbf{P}_k^d - \pi(Z))$ that is “good” for the proper closed subset $\pi(Z) \subset \mathbf{P}_k^d$ of pure dimension $d - 1$ (i.e., a hypersurface) in the sense of our toy model for curve fibrations in §6.2: the projection $\pi_\xi : \mathbf{P}_k^d - \{\xi\} \rightarrow \mathbf{P}_k^{d-1}$ away from ξ has (necessarily finite!) restriction $\pi(Z) \rightarrow \mathbf{P}^{d-1}$ that is generically étale.

Upon picking such a point ξ , we claim the non-empty finite set

$$S := \pi^{-1}(\xi) \subset X^{\text{sm}}$$

of k -points has the desired property that the blow-up of X along it admits a curve fibration over $\mathbf{P}^{d-1}$ whose fibers are all generically smooth. To understand this, we first use the explicit description from §6.2 of $\text{Bl}_\xi(\mathbf{P}^d)$ as a Zariski $\mathbf{P}^1$ -bundle over $\mathbf{P}^{d-1}$ (the space of lines through ξ in $\mathbf{P}^d$) to give an equally explicit description of $\text{Bl}_S(X)$. Consider the fiber product X' given by

$$\begin{array}{ccc} X' & \longrightarrow & \text{Bl}_\xi(\mathbf{P}^d) = \{(p, \ell) \in \mathbf{P}^d \times \mathbf{P}^{d-1} : p \in \ell\} \\ \downarrow \varphi & & \downarrow \\ X & \xrightarrow{\pi} & \mathbf{P}^d \end{array}$$

so X' is described functorially by

$$X' = \{(x, \ell) \in X \times \mathbf{P}^{d-1} : x \in \pi^{-1}(\ell)\}$$

In terms of this description, the map $\varphi : X' \rightarrow X$ given by projection onto the first coordinate is an isomorphism over $X - S = \pi^{-1}(\mathbf{P}^d - \{\xi\})$ since it is a base change of the analogous projection $\text{Bl}_\xi(\mathbf{P}^d) \rightarrow \mathbf{P}^d$ that is an isomorphism over $\mathbf{P}^d - \{\xi\}$.

We claim there is a unique X -isomorphism

$$X' \simeq \mathrm{Bl}_S(X)$$

(so the fiber-product construction of X' is an explicit description of the blow-up of X along S). Uniqueness is immediate because blow-ups have no nontrivial automorphism over the object being blown-up (due to the universal property of blow-up), so for existence we may work Zariski-locally over X . Over $X - S$ there's nothing to do since both X' and $\mathrm{Bl}_S(X)$ have structure maps to X that are isomorphisms over $X - S$. This makes our task Zariski-local around S , so it suffices to work over the open subset $\pi^{-1}(\Omega) \subset X$. But $\pi : \pi^{-1}(\Omega) \rightarrow \Omega$ is *flat* (even *étale*), so the compatibility of blow-up with flat base change reduces our task for (X, S) to the special case $(\mathbf{P}^d, \xi)$ that is just our old toy model.

With X' now known to be uniquely X -isomorphic to $\mathrm{Bl}_S(X)$, we can use the explicit description of X' as a fiber product to produce the desired curve fibration structure. Denote by

$$\begin{aligned} f : X' &\rightarrow \mathbf{P}^{d-1} \\ (x, \ell) &\mapsto \{\ell\} \end{aligned}$$

the projection onto the second coordinate. This will be the curve fibration that we desire. Note in particular that for $\{\ell\} \in \mathbf{P}^{d-1}(k)$ we have functorially $f^{-1}(\{\ell\}) = \{x \in X : x \in \pi^{-1}(\ell)\} \simeq \pi^{-1}(\ell)$. In other words, the k -point fibers of f (as schemes!) are *exactly* the preimages $\pi^{-1}(\ell)$ (as schemes!).

Since ℓ is locally cut out in $\mathbf{P}^d$ by $d - 1$ equations, the same goes for $\pi^{-1}(\ell)$ in the purely d -dimensional X , so $\pi^{-1}(\ell)$ has all irreducible components of dimension *at least* $d - (d - 1) = 1$ by Krull's Hauptidealsatz. But $\pi^{-1}(\ell) \rightarrow \ell$ is a finite surjection, so in fact the irreducible components of $\pi^{-1}(\ell)$ must all have dimension exactly 1. In other words, the fibers of $f : \mathrm{Bl}_S(X) = X' \rightarrow \mathbf{P}^{d-1}$ over k -points are all non-empty with pure dimension 1.

We can push this a bit further: since $\xi \in \Omega$ and ℓ passes through ξ in $\mathbf{P}^d$, the overlap $\ell \cap \Omega$ is a *non-empty* open subset of the line ℓ . Hence, the finite surjection $\pi^{-1}(\ell) \rightarrow \ell$ is *étale* over the dense open $\ell \cap \Omega$ whose open preimage in $\pi^{-1}(\ell)$ is dense for dimension reasons (all generic points in $\pi^{-1}(\ell)$ live over the generic point of ℓ). In other words, the fiber $f^{-1}(\{\ell\}) \simeq \pi^{-1}(\ell)$ contains a dense open subset $\pi^{-1}(\ell \cap \Omega)$ that is *étale* over the smooth $\ell \cap \Omega$ and hence is *smooth*! This shows that each fiber of f over a k -point has a dense smooth locus.

Since k is algebraically closed, standard results on generic flatness, “spreading out” principles, and constructibility of loci in the base for properties of geometric fibers (detailed with extraordinary thoroughness in [EGA, IV₃, IV₄], for example) tell us that *all* (geometric) fibers of f are non-empty of pure dimension 1 with dense smooth locus.

§9.2 Relating fibral smoothness to relative smoothness

Having shown that all fibers are generically smooth, we next want to show that in fact the open relative smooth locus $U := \text{sm}(X'/\mathbf{P}^{d-1}) \subset X' \subset X$ is fiberwise-dense. The issue is that we haven't analyzed where $X' \rightarrow \mathbf{P}^{d-1}$ is flat, so a-priori it isn't clear if a point smooth in a fiber X'_y is also a smooth point for the morphism $X' \rightarrow \mathbf{P}^{d-1}$ (the converse is automatic). We shall show that there is no confusion: anywhere that a fiber X'_y of $f : X' \rightarrow \mathbf{P}^{d-1}$ is smooth is also a smooth point for f . This will be checked for fibers over k -points, which is enough to prove it in general (by using spreading-out principles) since k is algebraically closed.

REMARK 9.2.1. For mere fiberwise-density of the smooth locus, rather than the more precise result promised above, one can argue in a more direct way as follows. It suffices to show $f : X' \rightarrow \mathbf{P}^{d-1}$ is smooth on the open set $\varphi^{-1}(\pi^{-1}(\Omega))$ (for $\varphi : X' \rightarrow X$ the blow-up map) since for $y = \{\ell\} \in \mathbf{P}^{d-1}(k)$ this meets the fiber $X'_y = f^{-1}(\{\ell\}) \simeq \pi^{-1}(\ell)$ over y in the open subset $\pi^{-1}(\ell \cap \Omega) \subset \pi^{-1}(\ell)$ that is dense.

But f factors as $X' \rightarrow \text{Bl}_\xi(\mathbf{P}^d) \rightarrow \mathbf{P}^{d-1}$ whose second step is a Zariski $\mathbf{P}^1$ -bundle (hence smooth), so it suffices to show that first map has étale restriction to

$$\varphi^{-1}(\pi^{-1}(\Omega)) = \pi^{-1}(\Omega) \times_\Omega \text{Bl}_\xi(\Omega).$$

This restriction is the composite map

$$\pi^{-1}(\Omega) \times_\Omega \text{Bl}_\xi(\Omega) \xrightarrow{\text{pr}_2} \text{Bl}_\xi(\Omega) \hookrightarrow \text{Bl}_\xi(\mathbf{P}^d)$$

whose second step is an open immersion and first step is a base change of $\pi : \pi^{-1}(\Omega) \rightarrow \Omega$ that is étale by how Ω was chosen.

This more direct approach is too limited for our purposes because we are going to want to control smoothness on *entire* fibers, and know that smoothness of an entire fiber implies smoothness of f on that entire fiber. The preceding argument does not yield that.

Let $Y = \mathbf{P}^{d-1}$. To show that the inclusion of open sets

$$X'_y \cap \text{sm}(X'/Y) \subset (X'_y)^{\text{sm}}$$

for $y \in Y(k)$ is an equality, it suffices to check on k -points since k is algebraically closed. In other words, for $x' \in (X'_y)^{\text{sm}}(k)$ we want to show that $f : X' \rightarrow Y$ is smooth at x' . Since x' and $y = f(x')$ have the same residue field k , one of the characterizations of smoothness in terms of completed local rings [EGA, IV₄, Prop. 17.5.3(d'')] the smoothness of f at x' is equivalent to the induced map

$$A := \mathcal{O}_{Y,y}^\wedge \rightarrow \mathcal{O}_{X',x'}^\wedge =: B$$

between completed local rings making the A -algebra B a formal power series ring over A . Since $\dim A = d - 1$ and $\dim B = d$ (as X and Y are varieties with dimensions d and $d - 1$ respectively), and the fibral smoothness at x' tells us that $B/\mathfrak{m}_A B = \mathcal{O}_{X'_y, x'}^\wedge \simeq k[[t]]$ as a k -algebra, we may conclude via the following result that is a slight weakening of [deJ96, p. 2.8].

LEMMA 9.2.2 (weak 2.8). Let $A \rightarrow B$ be a map between complete local noetherian rings with a common residue field k . Assume A is a domain and that the k -algebra $B_0 := B/\mathfrak{m}_A B$ is isomorphic to $k[[t_1, \dots, t_r]]$ where $r + \dim A = \dim B$. Then $B \simeq A[[t_1, \dots, t_r]]$ as an A -algebra.

Proof. Let $T_1, \dots, T_r \in \mathfrak{m}_B$ lift $t_1, \dots, t_r \in B_0$, so by completeness of B we have a unique local A -algebra map

$$g : A[[X_1, \dots, X_r]] \rightarrow B$$

satisfying $g(X_i) = T_i$ for all i . Our aim is to show that g is an isomorphism.

By design, $g_0 := g \bmod \mathfrak{m}_A$ is an isomorphism (coming from the given isomorphism between B_0 and $k[[t_1, \dots, t_r]]$ over k). By induction we claim each $g_n := g \bmod \mathfrak{m}_A^{n+1}$ is surjective. This is a standard successive approximation argument as follows. The evident commutative square expressing g_{n+1} “over” g_n reduces surjectivity for g_{n+1} (given it for g_n) to surjectivity for

$$\mathfrak{m}_A^n(A/\mathfrak{m}_A^{n+1})[[X_1, \dots, X_r]]/\mathfrak{m}_A^n(B/\mathfrak{m}_A^{n+1}B).$$

But the target only depends on elements of B through their reduction into B_0 (since $\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ is killed by $\mathfrak{m}_A$), and by hypothesis $k[[X_1, \dots, X_r]] \rightarrow B_0$ via $X_i \mapsto T_i$ is surjective. This completes the proof that every g_n is surjective.

Since B is $\mathfrak{m}_A$ -adically separated and complete (as it is so for the $\mathfrak{m}_B$ -adic topology), and $A[[X_1, \dots, X_r]]$ is as well, the *method* of proof of the surjectivity of every g_n gives even more: any element of B is a limit of the image of an $\mathfrak{m}_A$ -adic Cauchy sequence in $A[[X_1, \dots, X_r]]$. The limit of that Cauchy sequence provides the desired limit of any chosen element of B , so g is surjective. Letting $I = \ker(g)$, we have

$$A[[X_1, \dots, X_r]]/I \simeq B.$$

It now suffices to show I is zero. If I is not zero, say $F \in I$ is nonzero, then B is a quotient of

$$R := A[[X_1, \dots, X_r]]/(F).$$

But A is a *domain*, so that power series ring is a local noetherian *domain* of dimension $r + \dim A = \dim B$, so its quotient R by the nonzero F satisfies $\dim R < \dim B$. This contradicts that B is a quotient of R (so $\dim B \leq \dim R$). $\square$

§9.3 Genericity to ensure many smooth fibers

To ensure that some fiber X'_y is smooth (rather than just generically smooth) when X is normal, we need to be more attentive to the “genericity” of both ξ and the successive choices of points used to make the $N - d$ point projections whose composition defines the finite generically étale $\pi : X \rightarrow \mathbf{P}_k^d$. The target of $f : X' \rightarrow \mathbf{P}_k^{d-1}$ is the space of lines ℓ in $\mathbf{P}_k^d$ through ξ , and if $y \in \mathbf{P}^{d-1}(k)$ corresponds to ℓ then $f^{-1}(y) \simeq \pi^{-1}(\ell)$. If $d = N$ then $X = \mathbf{P}_k^N$ and there is nothing to do (for any $p \in \mathbf{P}^N(k)$ the blow-up $X' = \text{Bl}_p(X)$ is a Zariski $\mathbf{P}^1$ -bundle over the hyperplane $\mathbf{P}_k^{N-1}$ of lines through p). Hence, we now may and do assume $d \leq N - 1$.

The idea is firstly to show that $\pi^{-1}(\ell) = X \cap L$ as schemes for a linear subspace $L \subset \mathbf{P}_k^N$ of codimension $d - 1 \geq 1$ built from the choices underlying the $N - d$ successive point-projections, and then to check that the L ’s which arise in this way as we vary through all such choices allowed sweep out at least a *dense open* locus in $\text{Gr}(d - 1, N)$. Granting that, by Bertini’s Theorem $L \cap X^{\text{sm}}$ is smooth for a dense open locus of such L ’s and $L \cap (X - X^{\text{sm}})$ is empty for a dense open locus of such L ’s (since $X - X^{\text{sm}}$ has dimension at most $d - 2 < d - 1$, by normality of X). Hence, $L \cap X = L \cap X^{\text{sm}}$ is smooth for a dense open locus of such L ’s, so we would be done. This use of Bertini will entail some possibly additional “genericity” condition on ξ as chosen above (i.e., we’re not claiming all ξ as permitted previously will do the job).

Let’s now explain how to reinterpret the curve $\pi^{-1}(\ell) \subset X$ as a slice $X \cap L$ (scheme-theoretically) inside $\mathbf{P}_k^N$ for a linear space L of codimension $d - 1$ in $\mathbf{P}_k^N$. To build π , we choose a “generic” k -point $p_1 \in \mathbf{P}^N - X$ for which the projection $q_1 : X \subset \mathbf{P}_k^N - \{p_1\} \rightarrow \mathbf{P}_k^{N-1}$ away from p_1 is well-adapted to both X and Z (e.g., birational on both if $d \leq N - 2$, and generically étale on X and birational on Z if $d = N - 1$). If $d \leq N - 2$ then we make a “generic” choice of k -point $p_2 \in \mathbf{P}_k^{N-1}$ for which projection $q_2 : q_1(X) \subset \mathbf{P}_k^{N-1} - \{p_2\} \rightarrow \mathbf{P}_k^{N-2}$ away from p_2 is well-adapted to both $q_1(X)$ and $q_1(Z)$, and so on until we get a map q_{N-d} to $\mathbf{P}_k^d$ for which $\pi = q_{N-d} \circ \cdots \circ q_1 : X \rightarrow \mathbf{P}_k^d$.

Note that the projective spaces of successively lower dimensions in this process each depend on *previous choices* of points (each is a space of lines through a chosen point). Our concrete realization of the target of a point projection as a hyperplane in the target projective space from the preceding step thereby puts all of this inside the original $\mathbf{P}_k^N$, so we have really made “generic” choices of a k -point $p_1 \in \mathbf{P}_k^N$, a k -point p_2 in a choice of hyperplane $H_1 \subset \mathbf{P}_k^N$ missing p_1 , a k -point p_3 in choice of hyperplane $H_2 \subset H_1$ missing p_2 (this H_2 is a special type of codimension-2 linear subspace of $\mathbf{P}_k^N$ since it is contained in H_1), and so on until we pick a point p_{N-d} in a special type of codimension- $(N - d - 1)$ linear subspace of $\mathbf{P}_k^N$ avoiding p_{N-d-1} . These $N - d$ points projectively span a linear subspace Λ of dimension $N - d - 1$ (we get this “expected dimension” due to the avoidances in how we chose the successive points).

The design of Λ ensures that it *avoids* X entirely, so $X \subset \mathbf{P}_k^N - \Lambda$. The projection $\mathbf{P}_k^N - \Lambda \rightarrow \mathbf{P}_k^d$ away from Λ has target a fixed choice of d -plane $W \subset \mathbf{P}_k^N$ disjoint from the $(N - d - 1)$ -plane Λ , and it is described on k -points by $y \mapsto \text{span}(y, \Lambda) \cap W$. This restricts to a map $X \rightarrow \mathbf{P}_k^d$ that is

exactly π (check!).

Now we're ready to obtain the desired description of $\pi^{-1}(\ell)$ for a line ℓ in $\mathbf{P}_k^d$ through a fixed choice of “generic” k -point $p_{N-d+1} := \xi$ in the final $\mathbf{P}_k^d$ (which classifies lines through p_{N-d} in the special codimension- $(N-d-1)$ linear subspace of $\mathbf{P}_k^N$, or equivalently special $(d+1)$ -dimensional linear subspace, at the last step for building π). A choice of line ℓ in $\mathbf{P}_k^d$ through ξ amounts to a choice of point p_{N-d+2} in a choice of hyperplane $V \subset \mathbf{P}_k^d$ avoiding ξ . Since ℓ is the projective span of $p_{N-d} = \xi$ and p_{N-d+1} , the preimage $\pi^{-1}(\ell)$ as a *scheme* is exactly $X \cap L$ for the linear space

$$L = \text{span}(p_{N-d+1}, p_{N-d}, \Lambda) = \text{span}(p_{N-d+1}, n_{N-d}, \dots, p_1)$$

(with Λ of dimension $N-d-1$). This L has the “expected dimension” $2 + (N-d-1) = N-d+1 = N - (d-1)$; i.e., it has codimension $d-1$ in $\mathbf{P}_k^N$.

Having reinterpreted our overall construction as $X \cap L$ for $L \in \text{Gr}(d-1, N)(k)$, what remains is to prove that the L 's obtained through the successive “generic” k -point choices sweep out at least a dense open subset of $\text{Gr}(d-1, N)(k)$. The subtlety (at the level of rigorous proof, not geometric intuition) is that our procedure for building these L 's entails *successive* choices of points where each is “generic” in terms that *depend on* the preceding choices. That is why it is not immediately clear (though it must be true) that we really do sweep out at least a dense open locus of k -points in $\text{Gr}(d-1, N)$ in this way. Appendix B gives the rigorous proof of such “iterated genericity”.

The proof in that appendix entails removing all non-canonical choices of successive hyperplanes in the preceding geometric description. This removal of non-canonicity is needed to express the overall construction in a *relative* manner (informally: in families) so that it can be rigorously described *over* a Grassmannian and thereby be analyzed with a suitable incidence correspondence among universal structures to establish the desired “dense open” property. The details in Appendix B feel rather longer than should be needed; if you can provide a rigorous proof that is much shorter then please let me know (but first make sure your proof is really a proof and not a hand-wave; the burden is on the prover to make a complete argument).

Lecture 10

Stein factorization and the 3-point lemma

§10.1 Geometric connectedness of fibers

Last time we saw that as we vary through all choices underlying the construction of the curve fibration

$$f : X' = \mathrm{Bl}_S(X) \rightarrow Y := \mathbf{P}^{d-1} = \{\ell \ni \xi\}$$

(choices being the $N - d$ point projections as well as $\xi \in \mathbf{P}^d(k)$ in “good position” relative to the hypersurface $\pi(Z) \subset \mathbf{P}^d$), for k -points $y = \{\ell\} \in \mathbf{P}^{d-1}(k)$ the resulting fiber $f^{-1}(y) \simeq \pi^{-1}(\ell)$ is identified with the scheme-theoretic intersection $L \cap X$ for a codimension- $(d - 1)$ linear subspace L of $\mathbf{P}^d$ for which the corresponding points $L \in \mathrm{Gr}(d - 1, N)(k)$ sweep out at least a dense open locus in that Grassmannian.

The normality of X ensures $X - X^{\mathrm{sm}}$ has dimension at most $d - 2 < d - 1$, so by Bertini’s Theorem as we vary through a suitable dense open locus $U \subset \mathrm{Gr}(d - 1, N)$ the points $L \in U(k)$ have nice properties: $L \cap (X - X^{\mathrm{sm}}) = \emptyset$ (so $L \cap X = L \cap X^{\mathrm{sm}}$), and $L \cap X^{\mathrm{sm}}$ is a smooth and geometrically irreducible curve. Hence, by making suitable choices for the point projections and ξ , we can guarantee that there is a k -point $y_0 = \{\ell_0\} \in Y(k) = \mathbf{P}^{d-1}(k)$ for which the fiber $f^{-1}(y_0)$ is *smooth and connected*. As we saw last time, it follows that $f : X' \rightarrow Y$ is smooth at all points of X'_{y_0} ; i.e., the open subset $\mathrm{sm}(X'/Y) \subset X'$ contains X'_{y_0} . But then by properness of f it also follows (as we discussed last time) that there is a dense open $V \subset Y$ containing y_0 such that $f^{-1}(V) \rightarrow V$ is smooth. From this we aim to show that f is its own Stein factorization (i.e., $\mathcal{O}_Y \rightarrow f_*(\mathcal{O}_{X'})$ is an isomorphism) and hence *all* fibers of f are geometrically connected.

Since the $\mathcal{O}_Y$ -algebra $f_*(\mathcal{O}_{X'})$ is a coherent sheaf of domains over the non-empty opens in $Y = \mathbf{P}^{d-1}$ that is normal, the isomorphism property for $\mathcal{O}_Y \rightarrow f_*(\mathcal{O}_{X'})$ is sufficient to check on stalks at the generic point of Y . In particular, we can first localize at y_0 to focus on the curve fibration

$$X' \times_Y \mathrm{Spec}(\mathcal{O}_{Y, y_0}) \rightarrow \mathrm{Spec}(\mathcal{O}_{Y, y_0})$$

that is smooth and proper (as $y_0 \in V$ with V open in Y), and then conclude via:

PROPOSITION 10.1.1. Let R be a local noetherian ring and Z is a proper flat R -scheme whose special fiber Z_0 is geometrically reduced and geometrically connected. The natural map $\alpha : R \rightarrow \mathcal{O}_Z(Z)$ is an isomorphism.

We only need this result here for normal R and R -smooth Z , but the generality stated is very useful; e.g., it applies to geometrically reduced degenerating families such as for semi-stable curves as we’ll discuss later.

Proof. Letting k be the residue field of R and $\mathfrak{m}$ the maximal ideal, since Z_0 is a proper k -scheme that is geometrically reduced and geometrically connected the finite k -algebra $\mathcal{O}_{Z_0}(Z_0)$ must be k . Hence, the natural map of k -algebras

$$\varphi_{\mathfrak{m}}^0 : H^0(Z, \mathcal{O}_Z)/\mathfrak{m} \rightarrow H^0(Z_0, \mathcal{O}_{Z_0}) = k$$

is surjective via $1 \mapsto 1$. But $\mathcal{O}_Z$ is R -flat since Z is R -flat, so the theory of base change for coherent cohomology ensures that $\varphi_{\mathfrak{m}}^0$ is an isomorphism. In other words, the finite R -algebra $H^0(Z, \mathcal{O}_Z)$ viewed modulo $\mathfrak{m}$ is spanned by 1, so by Nakayama's Lemma the natural map

$$\alpha : R \rightarrow H^0(Z, \mathcal{O}_Z)$$

is *surjective*.

It remains to show α is injective. Picking $z_0 \in Z_0$, the composite map

$$R \xrightarrow{\alpha} H^0(Z, \mathcal{O}_Z) \rightarrow \mathcal{O}_{Z, z_0}$$

is flat (since Z is R -flat) and local (since $z_0 \in Z_0$), so it is faithfully flat and thus *injective*. That forces the first step α to be injective. $\square$

§10.2 Axiomatized setup and 3-point lemma

Upon renaming X' as X (which we may do, replacing Z with its preimage in X' : note that this preserves normality because the blow-up locus S consisted of smooth points, and blow-up at a point on a smooth k -scheme is smooth), we now summarize via a series of axioms the situation we have arrived at. In this axiomatization, we do not include normality of the $(d-1)$ -dimensional base Y of the curve fibration (this is $\mathbf{P}^{d-1}$ above) because such normality will interact poorly with some subsequent constructions. We also note that if the original field k has been algebraically closed then alterations beyond birational maps have not yet appeared.

SETUP 10.2.1. Our current situation may be summarized by the following properties (numbered consistently with [deJ96]).

- (i)-(iv) X is a projective variety of dimension $d \geq 2$ over an algebraically closed field k , and $Z \subsetneq X$ is the (non-empty) closed proper support of an effective Cartier divisor.
- (v) X is normal.
- (vi) There exists a k -map $f : X \rightarrow Y$, where Y is a projective variety – not assumed to be normal – of dimension $d-1$ with the following properties:
 - (a) All fibers of f are geometrically connected of pure dimension 1.
 - (b) The open relative smooth locus $\text{sm}(X/Y)$ meets each fiber X_y in a dense open.

- (c) There exists a dense open $U \subset Y$ such that $f^{-1}(U) \rightarrow U$ is smooth.
- (d) The map $f|_Z : Z \rightarrow Y$ is finite (hence surjective for dimension reasons) and generically étale.

Note that, despite our work up until now where Y has been $\mathbf{P}^{d-1}$, we are giving ourselves much more generality to work with in the above setup. Our next goal is to add the following additional property to the Setup:

- (vi)(e) The intersection $\text{sm}(X/Y) \cap Z$ meets each fiber X_y^{sm} in at least 3 points per irreducible component, for all geometric points $y \in Y$.

This property will be attained via the following lemma.

LEMMA 10.2.2 (3-Point Lemma, §4.13 of [deJ96]). Let $f : X \rightarrow Y$ be a map of projective varieties over an algebraically closed field k such that properties (vi)(a) and (vi)(b) from 10.2.1 hold. Then there exists an effective Cartier divisor $D \subsetneq X$ such that

- (i) $f : D \rightarrow Y$ is finite and generically étale.
- (ii) For all geometric points $y \in Y$, the intersection $\text{sm}(X/Y) \cap D$ meets X_y^{sm} in at least 3 points per irreducible component.

Note that once we have the above lemma, by our observation in Remark 6.1.2, we may replace Z with $Z \cup D$ (with the reduced structure) to add property (vi)(e) to the running setup.

Our proof strategy, which we begin in detail next time, is to build this Cartier D in stages (beginning with making it Y -finite). In order to fulfill condition (ii) of the 3-point lemma, we will make use of noetherian induction on Y (or quasi-compactness). We'll build D via hyperplane slices of X for a well-chosen projective embedding.

Let $\mathcal{L}$ be a very ample line bundle on X , and consider the associated projective embedding

$$i : X \hookrightarrow \mathbf{P} := \mathbf{P}(\Gamma(X, \mathcal{L}))$$

(described on k -points by $x \mapsto \ker(\text{ev}_x : \Gamma(X, \mathcal{L}) \rightarrow \mathcal{L}(x))$; recall $\mathbf{P}(V) := \text{Proj}(\text{Sym}(V))$ classifies hyperplanes in V rather than lines in V). Note that this is non-degenerate in the sense that $i(X)$ is not contained in any hyperplane of $\mathbf{P}$ (indeed, hyperplanes in $\mathbf{P}$ are $H_s := \mathbf{P}(\Gamma(X, \mathcal{L})/ks)$ for nonzero $s \in \Gamma(X, \mathcal{L})$, and $X \cap H_s$ is the zero-scheme of s in X , a proper closed subscheme since $s \neq 0$ in $\Gamma(X, \mathcal{L})$).

Let $\mathbf{P}^\vee = \mathbf{P}(\Gamma(X, \mathcal{L})^*)$ be the dual projective space; its k -points are hyperplanes in the dual of the space of sections, which are $(\Gamma(X, \mathcal{L})/\ell)^*$ for lines $\ell \subset \Gamma(X, \mathcal{L})$. Such lines correspond to projective hyperplanes $H \subset \mathbf{P}$ via $\ell \mapsto H_\ell := \mathbf{P}(\Gamma(X, \mathcal{L})/\ell) \subset \mathbf{P}$. More broadly, as the reader can check, the functor of points in $\mathbf{P}^\vee$ is the functor of projective hyperbundle subbundles in $\mathbf{P}$ or equivalently the functor of hyperplane subbundles of $\Gamma(X, \mathcal{L})$ (over k -schemes).

For any $H \in \mathbf{P}^\vee(k)$ the scheme $X \cap H$ is a Cartier divisor in X since the equation of H is locally nonzero on X (as X is non-degenerate in $\mathbf{P}$). For $X \cap H$ to be Y -finite is exactly the condition that the proper map $X \cap H \rightarrow Y$ is quasi-finite, which amounts to finiteness of its fibers over $Y(k)$ (since k is algebraically closed). But the fiber over $y \in Y(k)$ is $X_y \cap H$, so for this to *not* be k -finite amounts to $X_y \cap H$ containing some irreducible component of the purely 1-dimensional X_y . We seek H for which $X_y \cap H$ is never 1-dimensional (for $y \in Y(k)$), as then $X \cap H$ is Y -finite; informally this is akin to Bertini's theorem for slicing X being attentive to the fibration structure $X \rightarrow Y$.

Lecture 11

Finiteness aspect of 3-point lemma

§11.1 Interpreting finiteness via linear spans

The scheme $\mathbf{P}^\vee \times Y$ classifies pairs (H, y) , and at the level of k -valued points the “bad” pairs are those for which $X_y \cap H$ is 1-dimensional. We claim these correspond to the k -points of a closed subset T . Indeed, for the closed subscheme

$$S = \{(H, x) \in \mathbf{P}^\vee \times X : x \in H\}$$

the map

$$g : S \hookrightarrow \mathbf{P}^\vee \times X \xrightarrow{1 \times f} \mathbf{P}^\vee \times Y$$

has fiber over a k -point (H, y) given by $\{H\} \times (X_y \cap H)$. Hence, all fibers of g have dimension at most 1 and the “bad” (H, y) are those for which the fiber is 1-dimensional. By upper semi-continuity of fiber dimension, this locus in the base $\mathbf{P}^\vee \times Y$ of 1-dimensional fibers is *closed*: there exists a closed subset $T \subset \mathbf{P}^\vee \times Y$ whose k -points are these bad pairs (H, y) we want to avoid.

We want to show (at the cost of replacing $\mathcal{L}$ with a sufficiently high tensor power) that the proper map $\text{pr}_1 : T \rightarrow \mathbf{P}^\vee$ is *not* surjective, as then its image has a *non-empty* open complement, so there exists $H_0 \in \mathbf{P}^\vee(k)$ (in fact, even a dense open locus of such points) for which (H_0, y) is “good” for all $y \in Y(k)$: $X_y \cap H_0$ is finite for all such y . Indeed, then $X \cap H \rightarrow Y$ has finite fibers over $Y(k)$ and thus is quasi-finite (as $k = \bar{k}$), so by properness it is finite as desired. To prove $\text{pr}_1(T) \neq Y$, it suffices to show that $\dim T < \dim \mathbf{P}^\vee$. This will be attained by replacing $\mathcal{L}$ with a sufficiently high tensor power (note that $\mathbf{P}$, $\mathbf{P}^\vee$, and T depend on $\mathcal{L}$, but $f : X \rightarrow Y$ does not).

To get an upper bound on $\dim T$ in terms of $\mathcal{L}$, we shall use the other projection map $\text{pr}_2 : T \rightarrow Y$. We seek a useful upper bound on the dimensions of fibers of T over $Y(k)$, as $\dim T$ is at most $\dim Y$ plus the maximum dimension of a fiber of $\text{pr}_2 : T \rightarrow Y$ over $Y(k)$ (a general fact about behavior of dimension for finite type k -schemes with algebraically closed k):

$$\dim T \leq \dim Y + \max_{y \in Y(k)} \dim(\text{pr}_2^{-1}(y)).$$

Let’s describe each fiber $\text{pr}_2^{-1}(y)$ explicitly to see it is controlled by the images of $\Gamma(X, \mathcal{L}) \rightarrow \Gamma(C, \mathcal{L}|_C)$ for irreducible component C of X_y . We have

$$\text{pr}_2^{-1}(y) = \bigcup_C \{H \in \mathbf{P}^\vee \mid H \supset i(C)\}$$

where the union runs over the irreducible components $C \subset X_y$ and $i : C \hookrightarrow X_y \hookrightarrow X \subset \mathbf{P}$ is the natural inclusion. Since H is a linear subspace, we have $H \supset i(C)$ if and only if $H \supset \text{span}(i(C)) =:$

Λ_C (the smallest linear space containing $i(C)$, which is to say the “projective linear span” of k -points in $i(C)$). For each $y \in Y(k)$ and irreducible component C of X_y , the condition $H \supset \Lambda_C$ on $\{H\} \in \mathbf{P}^\vee(k)$ amounts to $1 + \dim \Lambda_C$ independent linear conditions on H so $W_C := \{H \in \mathbf{P}^\vee \mid H \supset i(C)\}$ is a linear subspace of $\mathbf{P}^\vee$ with codimension $1 + \dim \Lambda_C$. Thus, we have

$$\begin{aligned} \dim T &\leq \dim Y + \max_{y \in Y(k)} \dim(\cup_{C \subset X_y} W_C) \\ &= \dim Y + \dim \mathbf{P}^\vee - \min_C (1 + \dim \Lambda_C) \end{aligned}$$

where C runs over all irreducible components of X_y for all $y \in Y(k)$.

We have reduced to showing $\dim Y - \min_C (1 + \dim \Lambda_C) < 0$, which is to say $1 + \dim \Lambda_C > \dim Y$ for all irreducible components C of all fibers X_y for $y \in Y(k)$. For any nonzero $s \in \Gamma(X, \mathcal{L}) - \{0\}$ and the hyperplane $H_s := \text{Proj}(\Gamma(X, \mathcal{L})/ks) \subset \mathbf{P}$, we have $i(C) \subset H_s$ if and only if the restriction $s|_C \in \Gamma(C, \mathcal{L}|_C)$ vanishes. Thus, Via the inclusion $i : C \subset X \subset \mathbf{P}(\Gamma(X, \mathcal{L}))$ we have $\Lambda_C = \mathbf{P}(\Gamma(X, \mathcal{L})/V_C)$ where $V_C := \ker(\Gamma(X, \mathcal{L}) \rightarrow \Gamma(C, \mathcal{L}|_C))$ and thus $1 + \dim \Lambda_C = \dim \Gamma(X, \mathcal{L})/V_C$. Since $\Gamma(X, \mathcal{L})/V_C = \text{im}(\Gamma(X, \mathcal{L}) \rightarrow \Gamma(C, \mathcal{L}|_C))$, we want $\dim \text{im}(\Gamma(X, \mathcal{L}) \rightarrow \Gamma(C, \mathcal{L}|_C)) \geq 1 + \dim Y$ for all irreducible components $C \subset X_y$ for all $y \in Y(k)$.

§11.2 Passing to a high tensor power

LEMMA 11.2.1. For a closed subvariety $X \hookrightarrow \mathbf{P}_k^M$ with dimension $d \geq 1$, $\mathcal{L} := \mathcal{O}(1)$, and a reduced irreducible closed curve $C \subset X$ the image of

$$\Gamma(X, \mathcal{L}^{\otimes n}) \rightarrow \Gamma(C, \mathcal{L}^{\otimes n}|_C)$$

has dimension $\geq n + 1$ for all $n \geq 1$.

Using $n = \dim Y$ in this lemma concludes the proof that by passing to a sufficiently high tensor power of $\mathcal{L}$ we can ensure that a dense open locus of points $H \in \mathbf{P}^\vee(k)$ satisfies that $X \cap H \rightarrow Y$ is finite.

Proof. Pick hyperplanes $H_1, H_2 \subset \mathbf{P}^M$ such that the non-empty (by Bezout) $H \cap C_1$ and $H \cap C_2$ are finite and disjoint; i.e. nonzero $s_1, s_2 \in \Gamma(\mathbf{P}^M, \mathcal{O}(1))$ such that the zero schemes $Z(s_i) \cap C = Z(s_i|_C)$ are finite non-empty and disjoint, so $s_1|_C, s_2|_C \in \Gamma(C, \mathcal{L}|_C)$ are linearly independent.

Let $V = ks_1|_C + ks_2|_C \subset \Gamma(C, \mathcal{L}|_C)$, which is 2-dimensional. It suffices to show $\text{Sym}^n(V) \rightarrow \Gamma(C, \mathcal{L}^{\otimes n}|_C)$ is injective since $\dim \text{Sym}^n(V) = n + 1$. Suppose $\sum_{i=0}^n a_i (s_1|_C)^{\otimes i} \cdot (s_2|_C)^{\otimes n-i} = 0$. “Dividing” by the nonzero $s_2|_C^n \in \Gamma(C, \mathcal{L}^{\otimes n}|_C)$ yields that $f := \frac{s_1|_C}{s_2|_C} \in k(C)$ satisfies $\sum_{i=0}^n a_i f^i = 0$ in $k(C)$. But $f \notin k$ since $s_1|_C$ and $s_2|_C$ are linearly independent over k . Thus, since k is algebraically closed, this forces $a_i = 0$ for all i ; that is the desired injectivity. $\square$

Lecture 12

Finer structure of D and Z

§12.1 Completion of proof of 3-point lemma

Last time we passed to $\mathcal{L}^{\otimes n}$ for $n \geq \dim Y$ which implies that there exists a dense open locus $\Omega \subset \mathbf{P}^\vee = \{H \subset \mathbf{P}\}$ such that if $H \in \Omega(k)$ then we have Y -finiteness of D :

$$\begin{array}{ccc} D := X \cap H & \hookrightarrow & X \\ & \searrow \text{finite} & \downarrow f \\ & & Y \end{array}$$

In particular, for dimension reasons, each irreducible component D_i of D is finite onto Y .

Recall $\mathcal{U} := \text{sm}(X/Y) \subset X$ is fiberwise dense, i.e. $\mathcal{U}_{\bar{y}} \subset X_{\bar{y}}$ is dense for all $\bar{y} \in Y(k)$ (note also that $\mathcal{U}_{\bar{y}} \subset X_{\bar{y}}^{\text{sm}}$). We want:

- (1) $D \xrightarrow{\text{finite}} Y$ to be generically étale
- (2) for all $y \in Y(k)$, $D \cap \text{sm}(X/Y)$ meets each irreducible component of X_y in ≥ 3 points (which implies the same for all geometric fibers $X_{\bar{y}}$ for all $y \in Y$ since k is algebraically closed).

Pick $y_1 \in Y(k)$. We seek a hyperplane H_1 such that for $D_1 = X \cap H_1$, we have $\text{Ét}(D_1/Y) \supset (D_1)_{y_1}$ and $D_1 \cap \text{sm}(X/Y)$ meets each irreducible component of X_{y_1} in ≥ 3 points. (We'll then see that automatically the same holds for all $y \in U_1(k)$ for some open $U_1 \ni y_1$ in Y .) The open subset $\text{sm}(X/Y)_{y_1} \subset X_{y_1}$ is *dense* by design, so consider H_1 which *misses* the finite set $X_{y_1} - \text{sm}(X/Y)_{y_1}$. Such “good” H_1 exhaust a dense open in $\mathbf{P}^\vee$, and imposing Y -finiteness on $D_1 := X \cap H_1$ is fulfilled by a dense open subset of possibilities for H_1 in $\mathbf{P}^\vee$; in such cases $(D_1)_{y_1} \subset \text{sm}(X/Y)_{y_1} \subset X_{y_1}^{\text{sm}}$.

Every irreducible component C of X_{y_1} *meets* H_1 by Bezout's Theorem (since C is a closed curve in $\mathbf{P}$), and by Bertini applied to the quasi-projective smooth curve $\text{sm}(X/Y)_{y_1}$ implies that a dense open locus of H_1 's have the additional property that $H_1 \cap \text{sm}(X/Y)_{y_1}$ is étale. As long as $H_1 \cap C$ has degree ≥ 3 we'll then have (2) for $D_1 := X \cap H_1$ at the points over $y = y_1$ by ensuring (as we may) that when we passed to $\mathcal{L}^{\otimes n}$ for a large n then we ensure $n \geq 3$.

We have arranged that the y_1 -fiber of $D_1 = X \cap H_1$ is étale (and even contained inside $\text{sm}(X/Y)_{y_1}$). Let's show that such fibral étaleness implies the morphism $D_1 \rightarrow Y$ is étale at all points over y_1 . This cannot be shown using Lemma 9.2.2 because the completed local rings on Y may not be domains (we dropped the normality condition on Y for the sake of later inductive purposes), but we have another property now that was not in hand when using Lemma 9.2.2 earlier: the points on $(D_1)_{y_1}$ are contained in the open subset $\text{sm}(X/Y)$ where $X \rightarrow Y$ is smooth. This allows us to deduce étaleness of $D_1 \rightarrow Y$ at the points over y_1 via:

LEMMA 12.1.1. Given

$$\begin{array}{ccc} X & \longleftrightarrow & D \\ f \downarrow & \swarrow & \\ Y & & \end{array} \quad \begin{array}{c} x \\ \downarrow \\ y \end{array}$$

where x, y are k -points, and X is of relative dimension 1 over Y , if we assume f is *smooth* at x and D_y is étale at x then $D \rightarrow Y$ is étale at x .

Caution: We cannot use Lemma 9.2.2 because $\hat{\mathcal{O}}_{Y,y}$ may not be a domain. The present result is a special case of the Jacobi Criterion for smoothness of a closed subscheme as stated in [BLR90, 2.2/7, (a) $\Leftrightarrow$ (d)] (with $r = 1$ in the notation therein).

Proof. We have $\hat{\mathcal{O}}_{X,x} \simeq \hat{\mathcal{O}}_{Y,y}[[t]]$ as $\hat{\mathcal{O}}_{Y,y}$ -algebras since X is Y -smooth at x with relative dimension 1 and $\kappa(x) = \kappa(y)$. The desired étaleness is equivalent to showing that the natural map $\hat{\mathcal{O}}_{Y,y} \rightarrow \hat{\mathcal{O}}_{D,x}$ is an isomorphism.

We have that $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/(h)$ for some nonzero $h \in \mathfrak{m}_x$, so

$$\hat{\mathcal{O}}_{D,x} = \hat{\mathcal{O}}_{X,x}/(h) \simeq \hat{\mathcal{O}}_{Y,y}[[t]]/(h)$$

and want its structure map from $\hat{\mathcal{O}}_{Y,y}$ to be an isomorphism. We have

$$k = \mathcal{O}_{D_y,x} = \hat{\mathcal{O}}_{D_y,x} \simeq \hat{\mathcal{O}}_{D,x}/\mathfrak{m}_y \simeq k[[t]]/(\bar{h})$$

as k -algebras where the first equality follows from étaleness and $\bar{h} := h \bmod \mathfrak{m}_y \in \mathcal{O}_{X_y,x}$. In particular $h \in (\text{unit})t + \mathfrak{m}_y[[t]]$ in $\hat{\mathcal{O}}_{Y,y}[[t]]$, since $\bar{h} = t(\text{unit})$ in $k[[t]]$ and thus $h \in (\text{unit})t + \hat{\mathfrak{m}}_y[[t]]$ in $\hat{\mathcal{O}}_{Y,y}[[t]]$. This implies (see Exercise 12.1.A below) that $\hat{\mathcal{O}}_{Y,y}[[x]] \rightarrow \hat{\mathcal{O}}_{Y,y}[[t]]$ defined by $x \mapsto h$ is an isomorphism. Thus, $\hat{\mathcal{O}}_{Y,y} \xrightarrow{\sim} \hat{\mathcal{O}}_{Y,y}[[t]]/(h)$. $\square$

EXERCISE 12.1.A. With everything as in the lemma above, prove the map $\hat{\mathcal{O}}_{Y,y}[[x]] \rightarrow \hat{\mathcal{O}}_{Y,y}[[t]]$ defined by $x \mapsto h$ is an isomorphism.

We now have

$$\begin{array}{ccc} D_1 & \hookrightarrow & X \\ & \searrow \text{finite} & \downarrow \\ & & Y \end{array}$$

such that the open subset $\text{Ét}(D_1/Y) \subset D_1$ contains $(D_1)_{y_1}$ and the open subset $D_1 \cap \text{sm}(X/Y) \subset D_1$ contains $(D_1)_{y_1}$. For any closed (e.g., finite or proper) $\varphi : Z \rightarrow Y$ and open $V \subset Z$ containing a fiber Z_y necessarily $V \supset \varphi^{-1}(\mathcal{U}_1)$ for some open $\mathcal{U}_1 \ni y_1$ in Y . Therefore there exists some open $\mathcal{U}_1 \subset Y$ around y_1 such that $(D_1)_{\mathcal{U}_1}$ is $\mathcal{U}_1$ -étale and $(D_1)_{\mathcal{U}_1} \subset \text{sm}(X_{\mathcal{U}_1}/\mathcal{U}_1)$. In particular, for all

$y \in \mathcal{U}_1(k)$ we have that $(D_1)_y \cap \text{sm}(X/Y)_y$ meets every irreducible component of X_y in at least 3 points.

If $\mathcal{U}_1 = Y$, we are done! If not, pick $y_2 \in (Y - \mathcal{U}_1)(k)$ to get $D_2 = H_2 \cap X$ where we ensure H_2 does not contain the generic points of D_1 . The closed subscheme $D_1 + D_2 \subset X$ (defined by the invertible ideal sheaf $\mathcal{I}_{D_1} \mathcal{I}_{D_2}$) has a dense open as a scheme that is dense and open in $D_1 \amalg D_2$, so the finite $D_1 + D_2 \rightarrow Y$ is also generically étale. This solves the problem over $\mathcal{U}_1 \cup \mathcal{U}_2$. We can continue in this fashion, and by the quasi-compactness of Y (or noetherian induction) we get finitely many $\mathcal{U}_i$ that cover Y . Using $D = \sum D_i$ finishes the proof of the 3-point Lemma.

§12.2 Effect of alteration on Y

Now by replacing Z with $(Z \cup D)_{\text{red}}$ (still the support of an effective Cartier divisor) we can also enforce a further property:

(vi)(e) : The overlap $\text{sm}(X/Y) \cap Z$ meets each geometric fiber $X_{\bar{y}}$ (necessarily inside $X_{\bar{y}}^{\text{sm}}$!) in at least 3 points on every irreducible component.

Let's now discuss how the array of properties we have arranged interact with applying an alteration to Y . To that end, consider the following geometric setup over an algebraically closed field k . We have a map $f : X \rightarrow Y$ between projective varieties with $\dim X = d \geq 2$, $Z \subset X$ is a non-empty closed subset that is the support of a Cartier divisor (so Z has pure dimension $d - 1$), and:

- (v) X is normal,
- (vi)(a-c) f has all fibers geometrically connected of pure dimension 1 (so $\dim Y = d - 1 > 0$) with the Zariski-open subset $\text{sm}(X/Y) \subset X$ fiberwise dense over Y (i.e., meets each X_y in a dense open subset), and there is a dense open $U \subset Y$ such that $X_U \rightarrow U$ is smooth,
- (vi)(d) $f : Z \rightarrow Y$ is finite and generically étale,
- (vi)(e) for all geometric points $\bar{y}$ of Y , the relative smooth locus $\text{sm}(X/Y)$ meets $Z_{\bar{y}}$ in at least 3 points on every irreducible component of $X_{\bar{y}}$.

Keep in mind that we do *not* make any assumptions about normality or especially smoothness for Y (even though in the original context it was a projective space), because we are going to need to replace Y with various alterations later on.

We wish to address the effect of “replacing” Y with another projective variety Y' that is a (generically étale) alteration of Y . We shall need to make such changes in Y at many stages in subsequent arguments and so must be clear at the outset about how this affects X and Z and the above running hypotheses. Though (v) was important to arrive at the present setup, that has served its purpose and it won't be a problem to give up (v) later on. Thus, our aim is really to

explain how all of the above properties *apart from* (v) are preserved under a suitable replacement process when given a generically étale alteration $\psi : Y' \rightarrow Y$ and trying to replace Y with Y' .

Let $X' = (X \times_Y Y')_{\text{red}}$, and $Z' = (Z \times_Y Y')_{\text{red}}$. We claim that $f' : X' \rightarrow Y'$ and $Z' \subset X'$ satisfy all of the above conditions except possibly that (v). Since $X_U \rightarrow U$ is smooth with geometrically connected fibers of dimension 1, for the dense open $U' = \psi^{-1}(U) \subset Y'$ clearly $X'_{U'} = X \times_Y U' = X_U \times_U U'$ is U' -smooth with geometrically connected fibers of dimension 1, so for the projective variety Y' of dimension $d - 1$ we see that $X' \rightarrow Y'$ is smooth over the dense open $U' \subset Y'$ with geometric fibers over U' that are smooth and connected of dimension 1. In particular, once we show that the reduced X' is irreducible (hence integral) it follows that X' has dimension d and dominant onto X , so the preimage Z' of Z in X' is the support of a Cartier divisor (hence is of pure dimension $d - 1$). So we now prove:

PROPOSITION 12.2.1. The reduced projective k -scheme X' is irreducible.

Proof. The Zariski-dense open $\Omega = \text{sm}(X/Y) \subset X$ is fiberwise-dense over Y , so likewise for $\Omega' := \Omega \times_Y Y'$ inside $X \times_Y Y'$ over Y' . In particular, Ω' is reduced, so this is also an open subscheme of $X' := (X \times_Y Y')_{\text{red}}$ that is fiberwise-dense over Y' . It follows that any irreducible component of X' must meet Ω' , so it suffices to show that Ω' is irreducible.

Since $\Omega \rightarrow Y$ is flat, so is $\Omega' \rightarrow Y'$. Thus, all generic points of Ω' lie over the generic point η' of Y' . But the alteration $Y' \rightarrow Y$ has fiber η' over the generic point η of Y , so $\Omega'_{\eta'} = \Omega_{\eta} \times_{\eta} \eta'$. It therefore suffices to show that Ω_{η} is geometrically irreducible over η . But Ω_{η} is dense open in the η -scheme X_{η} that is a smooth and geometrically connected curve, so the desired geometric irreducibility follows (since smooth connected schemes over fields are irreducible). $\square$

Since the open subscheme $\text{sm}(X/Y) \times_Y Y' \subset X \times_Y Y'$ is Y' -smooth (hence reduced) and fiberwise-dense over Y' , it is also an open subscheme of $X' = (X \times_Y Y')_{\text{red}}$ that is Y' -smooth and fiberwise-dense over Y' . Thus, it is contained in the Zariski-open subset $\text{sm}(X'/Y') \subset X'$, so $\text{sm}(X'/Y')$ is also fiberwise-dense inside X' over Y' . It is now clear that (vi)(a-c) holds for $X' \rightarrow Y'$ (using $U' = \psi^{-1}(U)$ as introduced above).

To show that Z' with pure dimension $d - 1$ is finite and generically étale over Y' , we note that finiteness is equivalent to quasi-finiteness (as Z' and Y' are k -proper), and that in turn is a topological property easily seen since $Z' = (Z \times_Y Y')_{\text{red}}$. The generic étaleness for Z' over Y' is therefore equivalent to η' -étaleness of the generic fiber $Z'_{\eta'}$. This generic fiber is a localization of Z' , and localization commutes with the formation of nilradicals, so $Z'_{\eta'}$ is the underlying reduced scheme of $(Z \times_Y Y')_{\eta'} = Z_{\eta} \times_{\eta} \eta'$ (recall that $Y'_{\eta} = \eta'$ as *schemes*, since $Y' \rightarrow Y$ is a generically finite dominant map between varieties). But Z_{η} is η -étale by (vi)(d), so $Z_{\eta} \times_{\eta} \eta'$ is η' -étale and therefore reduced. In other words, $Z'_{\eta'} = Z_{\eta} \times_{\eta} \eta'$, so η' -étaleness of $Z'_{\eta'}$ follows from η -étaleness of Z_{η} . In other words, (vi)(d) holds for $Z' \rightarrow Y'$.

Finally, (vi)(e) holds for $(X' \rightarrow Y', Z')$ because we have seen that $\text{sm}(X/Y) \times_Y Y' \subset \text{sm}(X'/Y')$

and for any geometric point $\bar{y}'$ of Y' over a geometric point $\bar{y}$ of Y the closed subset $Z'_{\bar{y}'} \subset X'_{\bar{y}'}$ has the same underlying space as $Z_{\bar{y}} \times_{\bar{y}} \bar{y}' \subset X_{\bar{y}} \times_{\bar{y}} \bar{y}'$.

§12.3 Arranging Z to be a union of sections

Our next step (corresponding to [deJ96, §4.16]) is to arrange that Z is topologically a union of finitely many (possibly overlapping) sections $\sigma_j : Y \rightarrow X$. In this setting we have

$$\begin{array}{ccccc}
 Z & \hookrightarrow & X & \longleftarrow & X_U & \longleftarrow & Z_U & & Z_\eta = \coprod \eta_i \\
 & \searrow & \downarrow & & \downarrow & \swarrow & & \downarrow & \\
 & & Y & \xleftarrow{\text{dense open}} & U & & & \eta
 \end{array}
 \quad \begin{array}{l} \\ \\ \text{finite étale} \\ \\ \end{array}$$

where the η_i 's are the generic points of Z (since Z has pure dimension $d - 1 = \dim(Y)$ and Z is Y -finite) and $k(\eta_i)/k(\eta)$ is finite separable for all i .

Pick $K/k(Y)$ a Galois finite extension which splits all η_i/η . Use the generically étale finite alteration $Y' \rightarrow Y$ that is the normalization of Y in $\text{Spec}(K) \rightarrow \eta$. Passing to $X' := (X \times_Y Y')_{\text{red}} \rightarrow Y'$ and $Z' := (Z \times_Y Y')_{\text{red}}$ as in §12.2, this brings us to the case that Y is normal and $Z_\eta = \coprod \eta_i$ with $\eta_i \simeq \eta$ for all i . The closures Z_i of the η_i 's in X are the irreducible components of Z (equipped with reduced structure). Each $Z_i \rightarrow Y$ is finite between varieties of the same dimension and *birational* (since $\eta_i \simeq \eta$). But we have Y is normal, so each $Z_i \rightarrow Y$ must be an *isomorphism*; the inverses define sections $\sigma_i : Y \simeq Z_i \hookrightarrow X$ whose union (topologically) is Z as desired.

Lecture 13

Stable Marked Curves

§13.1 Definition and constructions

The notion of semistable curve over a field was introduced in Theorem 3.2.1 and Definition 3.2.2, where we saw that the residue field at each of its singularities is automatically (by real work, not by definition) separable over the base field. For a general scheme S , a *semi-stable curve* over S is a finitely presented flat map $X \rightarrow S$ whose (geometric) fibers are semi-stable curves. We are going to be interested in semi-stable curves equipped with some “marked points” to cut down on automorphismss of fibers (such as PGL_2 acting on $\mathbf{P}^1$, or an elliptic curve acting on its underlying curve via translations).

DEFINITION 13.1.1. Fix integers $g, n \geq 1$ such that $2g - 2 + n > 0$ (i.e., $g \geq 2$ with $n \geq 0$, or $g = 1$ with $n \geq 1$, or $g = 0$ with $n \geq 3$). A *stable genus- g curve with n marked points* over a scheme S is a proper flat and finitely presented map $f : \mathcal{C} \rightarrow S$ equipped with an ordered n -tuple of sections $\sigma_i \in \mathcal{C}(S)$ such that:

- each geometric fiber $\mathcal{C}_{\bar{s}}$ is a connected semi-stable curve, with $\sigma_i \in \mathcal{C}^{\mathrm{sm}}(S)$ for all i ,
- $h^1(\mathcal{C}_{\bar{s}}, \mathcal{O}) = g$ for all geometric points $\bar{s}$ of S ,
- each irreducible component Z of a geometric fiber $\mathcal{C}_{\bar{s}}$ such that $Z \simeq \mathbf{P}^1$, there are at least 3 “special points” on Z : a marking $\sigma_i(\bar{s})$ or a point where Z meets *another* irreducible component of $\mathcal{C}_{\bar{s}}$.

REMARK 13.1.2. Next time we’ll see the significance of the requirement $2g - 2 + n > 0$: it makes automorphism groups of geometric fibers finite, and it is related to ensuring a certain canonically associated invertible sheaf on $\mathcal{C}$ is fiberwise ample over S .

§13.2 Gluing

To construct interesting examples of stable curves with marked points we use gluing of schemes along closed sets and gluing of points on schemes together. In this subsection we define both notions and use them to provide examples of stable curves with marked points.

Let X and Y be ringed spaces and $Z \rightarrow X$, $Z \rightarrow Y$ be closed immersions of a locally ringed space Z . Define the *gluing of X and Y along Z* to be

$$X \coprod_Z Y = (|X| \coprod_Z |Y|, (i_X)_* \mathcal{O}_X \times_{(i_Z)_* \mathcal{O}_Z} (i_Y)^* \mathcal{O}_Y)$$

(with ring-theoretic sheaf fiber product) for i_X , i_Y , and i_Z the inclusions of $|X|$, $|Y|$, and $|Z|$ respectively into the topological gluing $|X| \amalg_{|Z|} |Y|$. This is readily checked to be a pushout in the category of ringed spaces, and likewise for locally ringed spaces when objects and inclusions are locally ringed.

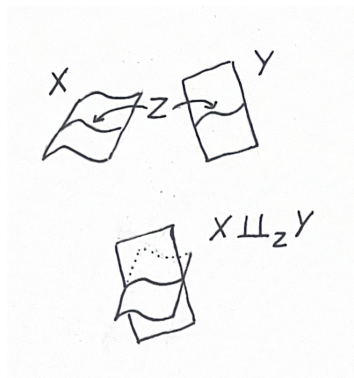


Figure 13.2.1: Two schemes glued along an isomorphism between closed subschemes

For affine $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ with a common closed subscheme $Z = \operatorname{Spec} C$ corresponding to surjections $f : A \twoheadrightarrow C$ and $g : B \twoheadrightarrow C$, consider the ring-theoretic fiber product

$$A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

There is a natural map

$$\operatorname{Spec} A \coprod_{\operatorname{Spec} C} \operatorname{Spec} B \rightarrow \operatorname{Spec}(A \times_C B)$$

defined by the universal property of affine schemes, and it is a pleasant exercise to check (please do, using that f and g are surjective) that this is an isomorphism. Building on this, one can show (please do) that if X , Y , and Z are schemes then there are compatible affine open covers of X , Y , and Z to ensure (via the settled all-affine case) that $X \amalg_Z Y$ is a scheme. This is called the *gluing of X and Y along Z* .

EXAMPLE 13.2.1. If A , B , and C are *finite type* R -algebras for noetherian R and we're given R -algebra surjections $A \twoheadrightarrow C \leftarrow B$ then the R -subalgebra $A \times_C B \subset A \times B$ is also finite type over R . Please check this, since in general even a polynomial ring $k[t_1, \dots, t_n]$ over a field can contain extremely nasty k -subalgebras (in particular not finitely generated).

In particular, the gluing of two finite type R -schemes along a closed subscheme is again finite type when R is noetherian.

We now discuss gluing together distinct rational points on a scheme over a field; this is particularly useful in the irreducible case (such as to make a nodal cubic by gluing distinct rational points in $\mathbf{P}^1$). First we focus on the affine case. Let k be a field and $X = \operatorname{Spec} A'$ be an affine scheme over k . Pick *distinct* k -points $P, Q \in X(k)$. We define the *gluing of X along P and Q* to be the

affine scheme $Y = \operatorname{Spec} A$ where A is the ring-theoretic pullback (with *surjective* right arrow, hence surjective left arrow)

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & \lrcorner & \downarrow (P,Q) \\ k & \xrightarrow{\Delta} & k \times k, \end{array}$$

i.e. $A = \{f \in A' \mid f(P) = f(Q)\}$. Note that $A \rightarrow A'$ is a module-finite k -algebra inclusion. A similar procedure works with k any ring and two k -valued points that are everywhere disjoint (ensuring $(P, Q) : A' \rightarrow k \times k$ is surjective).

It is an instructive exercise to check that this construction really is a ringed-space co-equalizer of $P, Q : \operatorname{Spec}(k) \rightrightarrows \operatorname{Spec}(A')$, as well as co-equalizer in the category of k -schemes.

At the level of completed local rings over a field k , it is also a good exercise to check (please do) that naturally

$$\hat{A}_R = k[[u]] \times_k k[[v]] = k[[u, v]]/(uv) \subset \hat{A}'_P \times \hat{A}'_Q = k[[u]] \times k[[v]].$$

EXAMPLE 13.2.2. Consider $A' = k[t]$, $P = 0$, and $Q = 1$. Then, $A = k[t(t-1), t^2(t-1)] \subset k[t]$ because (i) every integer $n \geq 2$ has the form $2a + 3b$ for non-negative integers a and b , (ii) this k -subalgebra contains polynomials of degree 2 and 3, (iii) a polynomial of degree at most 1 that vanishes at 0 and 1 is 0.

Letting $x = t(t-1)$ and $y = t^2(t-1)$, one checks $A = k[x, y]/(y^2 - x^3 - xy)$.

If we work over a field k with $\operatorname{char}(k) \neq 2$ then we could instead use $P, Q = \pm 1 \in k$, and then the corresponding gluing is $A = k[x, y]/(y^2 - x^2(x+1))$ for $x = t^2 - 1$ and $y = t(t^2 - 1)$.

The co-equalizer characterization allows us to bootstrap to carry out such a construction with any quasi-projective k -scheme via any choice of affine open containing the finitely many rational points being fused together.

One example of a stable marked curve is $\mathbf{P}^1$ with n glued pairs of distinct k -points as shown in Figure 13.2.2.

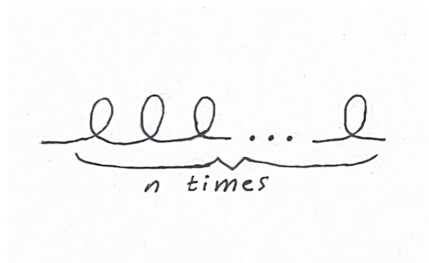


Figure 13.2.2: A projective line with n semi-stable self-gluing points for $n \geq 1$.

We next discuss how to compute the arithmetic genus of such self-gluing points. For this we'll use the *normalization sequence* as follows. Let C be a geometrically integral proper curve over a field

k with normalization $\tilde{C} \xrightarrow{\pi} C$. We have the exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \pi_* \mathcal{O}_{\tilde{C}} \rightarrow \mathcal{K} \rightarrow 0$$

for $\mathcal{K}$ a coherent (hence k -finite) skyscraper sheaf supported on the finite set of non-regular closed points of C . This gives the long exact sequence in cohomology

$$0 \rightarrow k \xrightarrow{\sim} k \rightarrow H^0(C, \mathcal{K}) \rightarrow H^1(C, \mathcal{O}_C) \rightarrow H^1(C, \pi_* \mathcal{O}_{\tilde{C}}) \rightarrow H^1(C, \mathcal{K}).$$

Since $\mathcal{K}$ is a skyscraper sheaf, we have $H^1(C, \mathcal{K}) = 0$. If C is semistable with k -rational singularities then $\mathcal{K} = \bigoplus_P (k \times k)/k$ where P runs over the singularities in C (to check this, consider the completed stalks $\mathcal{K}_P^\wedge$), and so $H^0(C, \mathcal{K}) = k^{\oplus m}$ for m the number of singularities of such C . Using these facts, when C is the self-gluing of $\mathbf{P}^1$ along n pairs of distinct k -points as in Figure 13.2.2 then $\tilde{C} \simeq \mathbf{P}^1$ and so the arithmetic genus g of C is n ; in other words, this is stable as a curve with *no marked points* if and only if $n \geq 2$.

Next we give some *reducible* examples. Let C_1 be $\mathbf{P}^1$ with two distinct points glued together; i.e. the curve in Figure 13.2.2 with $n = 1$. This appears in the first of the following two examples of stable marked curves with two marked points:

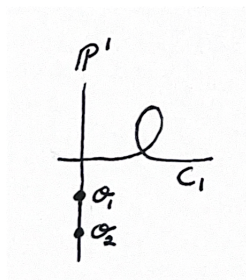


Figure 13.2.3: C_1 glued at a k -point to $\mathbf{P}^1$ with two marked points

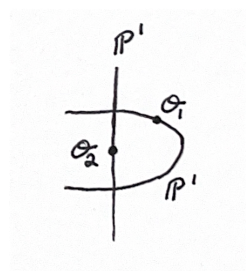


Figure 13.2.4: Two copies of $\mathbf{P}^1$ with a marked point, glued together at two distinct k -points

Let C be either of the curves in Figure 13.2.3 or Figure 13.2.4. Let C_1 be as defined for Figure 13.2.3 and either copy of $\mathbf{P}^1$ for Figure 13.2.4. Then, $g(C) = 1 + g(C_1)$; now $h^0(\pi_* \mathcal{O}_{\tilde{C}}) = 2$.

EXERCISE 13.2.A. Show that over an algebraically closed field k :

- (i) the only semistable curves with genus 0 are the trees of $\mathbf{P}^1$'s as shown in Figure 13.2.5; i.e. gluings of $\mathbf{P}^1$'s along k -points with incidence graph a tree (no loops);
- (ii) the only stable genus 0 curve with 3 markings is $(\mathbf{P}^1, \{0, 1, \infty\})$;
- (iii) the only stable curves with genus 1 and 1 marking are elliptic curves and the nodal cubic with one marking;
- (iv) the only *reducible* stable curves with genus 1 and $n > 1$ markings are the “loops of $\mathbf{P}^1$'s” as shown in Figure 13.2.6, along with such marked curves having trees of sufficiently marked $\mathbf{P}^1$'s attached at smooth k -points;
- (v) for a stable marked curve C , any irreducible component C' with $h^1(C', \mathcal{O}) = 1$ has at least 1 special point (i.e., a marking or meeting *another* irreducible component); this is a companion to the $\mathbf{P}^1$ -condition in the definition of stability of a semistable curve.

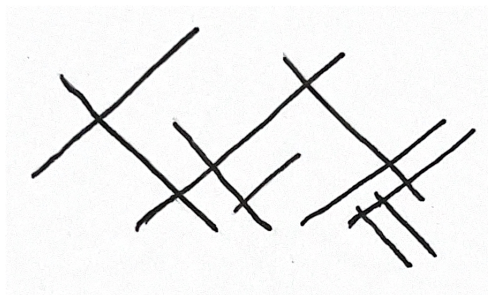


Figure 13.2.5: A tree of $\mathbf{P}^1$'s

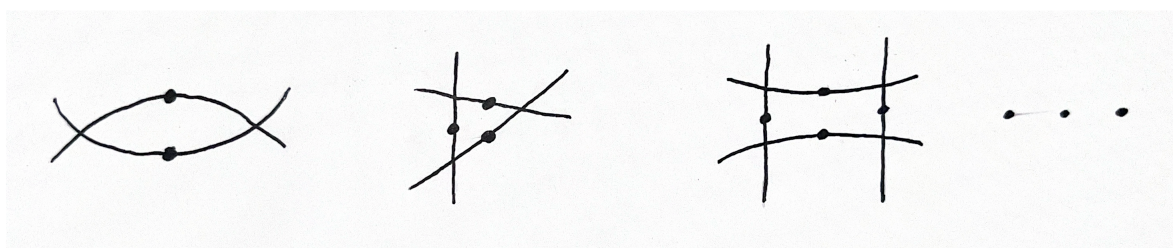


Figure 13.2.6: Loops of marked $\mathbf{P}^1$'s

§13.3 Relative ingredients

The main references on the relative theory of stable marked curves are [DM69, §1] (for $g \geq 2$ with $n = 0$) and [Knu83] (for general (g, n)), and rely on extensive use of coherent duality (over a general base, not just over fields). The construction of moduli “spaces” – in the sense of schemes, algebraic

spaces, or stacks – for moduli problems with semistable curves ultimately rests on Hilbert schemes (introduced in §C.1), and will be a topic of some discussion later on. All such procedures begin by applying the following result to universal families over Hilbert schemes:

LEMMA 13.3.1. For a proper fppf map $f : X \rightarrow S$, the subset

$$U = \{s \in S : X_{\bar{s}} \text{ is a semistable connected curve}\} \subset S$$

is open

Proof. The openness in S of the locus of points s for which X_s has a specific pure dimension (such as 1) is [EGA, IV₃, 12.2.1(ii)], and the locus of points s for which X_s is geometrically connected and geometrically reduced is open by [EGA, IV₃, 12.2.4(vi)]. Over the resulting open subset $V \subset S$ for which the geometric fibers are reduced and connected curves (so the open subset $\text{sm}(X_V/V) \subset X_V$ is fiberwise dense), the locus of points $s \in V$ for which $X_{\bar{s}}$ has all of its finitely many singularities semistable is also open due to the relative étale-local structure theorem for “ordinary double-point singularities” (applied in relative dimension 1) as stated and proved in [FK88, Ch. III, Prop. 2.7] (who proof uses Artin approximation, a tool we’ll use later in the course for others purposes and discuss in the final lecture). $\square$

Lecture 14

Stability and Ampleness

§14.1 Characterizations of stability

The stability condition involving $\mathbf{P}^1$ in geometric fibers in Definition 13.1.1 may seem a bit out-of-thin-air when first encountered. To demystify it, we now record two lemmas that provide rather different-looking alternative characterizations. For clarity, we state these results *without* requiring $2g - 2 + n > 0$ so that this can also shed some light on where that positivity condition comes from.

LEMMA 14.1.1. Let X be a proper connected semistable curve over a field $k = \bar{k}$ with marked points $\sigma_1, \dots, \sigma_n \in X^{\text{sm}}(k)$. The following are equivalent:

1. $(X; \sigma_1, \dots, \sigma_n)_{k[\varepsilon]}$ has no nontrivial automorphism over $k[\varepsilon]$ lifting the identity over k ,
2. The locally finite type k -group $\underline{\text{Aut}}_{(X; \sigma_1, \dots, \sigma_n)/k}$ is étale,
3. The abstract group $\text{Aut}(X; \sigma_1, \dots, \sigma_n)$ is finite.
4. The pair of simultaneous conditions (i) every irreducible component Z of X with $Z \simeq \mathbf{P}^1$ has at least 3 special points, and (ii) every (reduced) irreducible component C of X with arithmetic genus $h^1(C, \mathcal{O}_C) = 1$ has at least one special point.

In condition 4, we recall that “special point” means any σ_i or a point on two *distinct* irreducible components (so the singularity on a nodal cubic is *not* “special”). By Exercise 13.2.A(v), condition 4(ii) above is automatic when $2g - 2 + n > 0$; that explains why this condition is not mentioned in [DM69] (where $g \geq 2$ and $n = 0$, so $2g - 2 + n > 0$) and more broadly it is usually not mentioned in the definition of stability in the literature, where one is usually assuming $2g - 2 + n > 0$ from the outset.

Proof. We give a sketch of $(4) \Rightarrow (1)$; the rest is easier (e.g., the equivalence of (1) and (2) is the infinitesimal étaleness criterion, which for a locally finite type k -group is sufficient to check at the identity). Suppose that ϕ is a $k[\varepsilon]$ -automorphism of $X_{k[\varepsilon]}$ lifting the identity on X over k . Any such ϕ is topologically the identity map (as that can be checked modulo ε), so it amounts to a $k[\varepsilon]$ -algebra automorphism of $\mathcal{O}_X[\varepsilon]$ of the form $f + g\varepsilon \mapsto f + (g + D(f))\varepsilon$ for $D : \mathcal{O}_X \simeq \mathcal{O}_X$ a k -linear derivation. By the universal property of the sheaf of relative 1-forms, such D uniquely factors as $\mathcal{O}_X \xrightarrow{d} \Omega_{X/k}^1 \xrightarrow{\delta} \mathcal{O}_X$ for an $\mathcal{O}_X$ -linear map $\delta : \Omega_{X/k}^1 \rightarrow \mathcal{O}_X$. Beware that we allow X to be singular (that is the entire purpose of the concept of stability in the first place, ultimately to allow a controlled class of singular objects to permit the valuative criterion of properness to be satisfied by the corresponding moduli “space”), so the coherent sheaf $\Omega_{X/k}^1$ on the reduced generically smooth X is generally not invertible.

We want to show that the requirement $\phi(\sigma_i) = \sigma_i$ in $X(k[\epsilon])$ for all i forces ϕ to be the identity map, which is to say $D = 0$ or equivalently $\delta = 0$. Since X is reduced, the vanishing of the $\mathcal{O}_X$ -valued δ is equivalent to the vanishing of its restriction to the dense open X^{sm} . That restriction $\delta|_{X^{\text{sm}}}$ corresponds to an algebraic vector field $\vec{v}_\delta$ on X^{sm} . One checks that $\phi \circ \sigma_i = \sigma_i$ over $k[\epsilon]$ if and only if $\vec{v}_\delta(\sigma_i) = 0$ in the tangent line $T_{\sigma_i}(X^{\text{sm}})$ over k , so we want to show that requiring this vector field to have a zero at all σ_i 's forces $\vec{v}_\delta = 0$.

As an illustration, when X is *smooth* with genus $g \geq 2$, so the properties in condition (4) are vacuously satisfied, this amounts to the classical fact that a smooth geometrically connected curve with genus $g \geq 2$ has no nonzero global vector field. That holds because the invertible dual sheaf $(\Omega_{X/k}^1)^\vee$ has degree $-(2g - 2) = 2 - 2g < 0$ and thus has no nonzero global sections. In general, the case $g \geq 2$ with $n = 0$ (allowing non-smooth X !) is [DM69, §1.4], which uses coherent duality on semistable curves over an algebraically closed field. One can adapt those arguments to general (g, n) . $\square$

§14.2 Relative ampleness and openness

Consider data $(X, \underline{\sigma}) := (X; \sigma_1, \dots, \sigma_n)$ as Lemma 14.1.1. We can attach to this several invertible sheaves as follows. Firstly, each k -point σ_i belongs to the smooth locus X^{sm} , the corresponding ideal sheaf $\mathcal{I}_{\sigma_i} \subset \mathcal{O}_X$ (which coincides with the structure sheaf on $X - \{\sigma_i\}$ and in particular near the singularities) is invertible (as $\mathcal{O}_{X, \sigma_i}$ is a dvr). Although $\Omega_{X/k}^1$ is *not* invertible when X is not smooth, there is an invertible replacement via coherent duality as follows. The scheme X is Gorenstein since

$$\hat{\mathcal{O}}_{X, x} \simeq \begin{cases} k[[t]] & x \in X^{\text{sm}}(k), \\ k[[u, v]]/(uv) & x \notin X^{\text{sm}}(k), \end{cases}$$

so the relative “dualizing complex” for X over k (it makes sense for any finite type scheme over a regular ring, for example) is an invertible sheaf supported in degree -1 : it is $\omega_{X/k}[1]$ for an invertible sheaf $\omega_{X/k}$. (The Cohen-Macaulay property on X encodes that this complex is a coherent sheaf supported in a single degree, and the more refined Gorenstein condition on X encodes that this coherent sheaf is invertible.)

The theory of the relative dualizing complex is meaningful without properness, providing a robust theory for how to study the complex locally on X . In particular, even though X^{sm} is generally not proper, this theory provides a natural isomorphism $\omega_{X/k}|_{X^{\text{sm}}} \simeq \Omega_{X^{\text{sm}}/k}^1$. If $j : X^{\text{sm}} \hookrightarrow X$ is the natural open immersion then reducedness of X yields an injectivity of

$$\omega_{X/k} \rightarrow j_*(\omega_{X/k}|_{X^{\text{sm}}}) = j_*(\Omega_{X^{\text{sm}}/k}^1).$$

There is a classical recipe due to Rosenlicht that explicitly describes this invertible subsheaf $\omega_{X/k}$ in terms of the invertible sheaf of 1-forms on the *smooth* normalization $\pi : \widetilde{X} \rightarrow X$, a description that

is very useful when carrying out local calculations with $\omega_{X/k}$: if $\Sigma \subset \widetilde{X}$ is the set of k -points over the singularities on X (recall $k = \bar{k}$) then $\omega_{X/k}$ is the coherent subsheaf of $\pi_*(\Omega_{\widetilde{X}/k}^1(S))$ consisting of rational 1-forms on $\widetilde{X}$ with at worst simple poles along S and residues summing to 0 over each singularity on X .

As with relative 1-forms in general, the formation of $\omega_{X/k}$ commutes with étale localization on X ; i.e., for $f : U \rightarrow X$ an étale morphism over k there is a natural isomorphism $f^*\omega_{X/k} \simeq \omega_{U/k}$, so $\omega_{X/k}$ can be “computed” near singularities by using U that is étale over $\{uv = 0\} \subset \mathbf{A}_k^2$. We define the *invertible* sheaf

$$\tilde{\omega}_{X/k} := \omega_{X/k} \otimes \mathcal{I}_{\sigma_1}^\vee \otimes \dots \otimes \mathcal{I}_{\sigma_n}^\vee.$$

(This notation on the left side is a bit imprecise since it does not record the dependence on the marked points, but we don’t want to overload the notation.) To make contact with $2g - 2 + n$ we need to discuss a notion of “degree” for the invertible $\tilde{\omega}_{X/k}$ on the possibly non-regular X , so we now briefly digress to discuss this.

On regular proper curves C over a field F , we can express invertible sheaves $\mathcal{L}$ on C as $\mathcal{O}_C(D)$ for a Weil divisor $D = \sum n_i P_i$ (for closed points P_i) determined up to linear equivalence, and thereby use the notion of F -degree $\deg_F(D) = \sum n_i \deg_F(P)$. For reduced (hence generically regular) C there is a good relation between invertible sheaves and Weil divisors supported in the dense open regular locus, and this permits a definition of $\deg(\mathcal{L})$ by using F -degree for Weil divisors. This topic allowing non-regular C is beautifully developed from scratch in [BLR90, §9.1], where it is shown that $\deg(\mathcal{L})$ can also be defined more conceptually with good properties via Euler characteristic: for invertible $\mathcal{L}$ on a proper F -scheme X with pure dimension 1,

$$\chi(\mathcal{L}^{\otimes m}) := \dim_F H^0(C, \mathcal{L}^{\otimes m}) - \dim_F H^1(C, \mathcal{L}^{\otimes m})$$

is always a polynomial in $m \in \mathbf{Z}$ with integer coefficients and degree at most 1, with the linear coefficient may be *defined* to be the degree: this Euler characteristic is $(\deg \mathcal{L})m + c$ for the constant $c = \chi(\mathcal{O}_C)$. Among the good properties are:

- $\deg(\mathcal{L}_1 \otimes \mathcal{L}_2) = \deg \mathcal{L}_1 + \deg \mathcal{L}_2$ for invertible $\mathcal{L}_1, \mathcal{L}_2$ on C ;
- $\deg(\mathcal{I}_P) = -1$ for regular points $P \in X(F)$ (so $\deg(\mathcal{I}_P^\vee) = 1$);
- if $\{C_i\}$ is the set of irreducible components of C equipped with the reduced structure then $\deg(\mathcal{L}) = \sum \deg(\mathcal{L}|_{C_i})$.

Returning to our proper connected semistable X over an algebraically closed field k , coherent duality yields $\deg(\omega_{X/k}) = 2g - 2$ for $g := h^1(X, \mathcal{O}_X)$ (this is proved by a variant of the classical arguments using Riemann-Roch when X is smooth). Since $\deg \mathcal{I}_{\sigma_i} = -1$ for all i , we then have

$$\deg(\tilde{\omega}_{X/k}) = 2g - 2 + n.$$

Now we can finally state the second characterization of the stability condition on a semistable marked curve and use it to interpret “ $2g - 2 + n > 0$ ” is a more conceptual way:

LEMMA 14.2.1. For $(X, \underline{\sigma})$ as in Lemma 14.1.1, the following are equivalent:

1. condition (4) in Lemma 14.1.1,
2. $\tilde{\omega}_{X/k}$ is ample.

Proof. The case $g \geq 2$ with $n = 0$ is given in [DM69, Thm. 1.2], which uses coherent duality. This argument adapts to $g \geq 2$ with any n , as well as $g \leq 1$ with $n \geq 4$. For $g \leq 1$ with $n \leq 3$, see [Knu83, Cor.1.10]. $\square$

As with proper k -schemes in general, ampleness of an invertible sheaf on X is equivalent to ampleness of its restriction to each (reduced) irreducible component X_i of X . Moreover, on an *irreducible* reduced proper curve over a field the ampleness of an invertible sheaf is equivalent to *positivity* of its degree, so since

$$2g - 2 + n = \deg(\tilde{\omega}_{X/k}) = \sum \deg(\tilde{\omega}_{X/k}|_{X_i})$$

we see by Lemma 14.2.1 that the stability conditions in Lemma 14.1.1(4) *imply* $2g - 2 + n > 0$. The converse is generally false when X is reducible since positivity of $\sum \deg(\tilde{\omega}_{X/k}|_{X_i})$ does *not* force positivity of all summands. Nonetheless, the ampleness property provides a conceptual context for the requirement “ $2g - 2 + n > 0$ ” and it also gives a characterization of the stability conditions that makes sense when k is allowed to be an arbitrary field (and even paves the way to relativization over any scheme, as we’ll soon see).

COROLLARY 14.2.2. Fix g and n such that $2g - 2 + n > 0$. Consider a proper fppf map $f : X \rightarrow S$ with n sections $\sigma_1, \dots, \sigma_n \in X(S)$. The set

$$S^{\text{st}} := \{s \in S \mid (X_{\bar{s}}; \sigma_1(\bar{s}), \dots, \sigma_n(\bar{s})) \text{ is a stable } n\text{-pointed genus } g \text{ curve}\}$$

is open in S .

The hypothesis $2g - 2 + n > 0$ is harmless because if it doesn’t hold then S^{st} is empty and so the assertion would be devoid of interest and utility.

Proof. Last time we show that requiring the geometric fiber to be a connected semistable curve is an open condition on the base, as is the requirement $\sigma_i \in \text{sm}(X/S)$ (i.e., σ_i landing inside the open relative smooth locus or equivalently avoiding its closed complement in X). For $i \neq j$ the set $(\sigma_i \times \sigma_j)^{-1}(\Delta_{X/S}) \subset S$ of s for which $\sigma_i(s) = \sigma_j(s)$ is closed, so disjointness among the σ_i ’s is also an open condition on S . Hence, we now may and do pass to an open subset of X to arrange that

$X \rightarrow S$ has geometrically connected semistable fibers and the σ_i 's are pairwise disjoint sections all supported in the open relative smooth locus.

We have to show that the locus of such fibers which satisfy the stability condition is open. Just from the explicit formulation of the stability condition on geometric fibers (with $2g - 2 + n > 0$), this isn't at all apparent. To handle it, we are going to use Lemma 14.2.1 to recast the condition in terms of ampleness of $\tilde{\omega}_{X_s/k(s)}$. However, to do this requires being able to “interpolate” such fibral constructions, particularly the relative dualizing sheaf $\omega_{X_s/k(s)}$ provided by coherent duality over fields.

Now comes the atom bomb: by *relative* coherent duality, as for any flat and finitely presented S -scheme with Gorenstein fibers, we have a canonically associated invertible sheaf $\omega_{X/S}$ on X whose formation commutes with (i) any base change on S and (ii) any étale localization on X (i.e., if $f : U \rightarrow X$ is étale then naturally $f^*(\omega_{X/S}) \simeq \omega_{U/S}$). The quasi-coherent ideal sheaf $\mathcal{I}_{\sigma_i} \subset \mathcal{O}_X$ of each section is *invertible*, as we can check by working Zariski-locally on X to pass to where (X, σ_i) is étale over $(\mathbf{A}_S^1, 0)$ and making a Zariski-local calculation. By S -flatness of the closed subscheme $\sigma_i : S \hookrightarrow X$ the formation of $\mathcal{I}_{\sigma_i}$ also commutes with any base change on S . Thus, the sheaf

$$\tilde{\omega}_{X/S} := \omega_{X/S} \otimes \mathcal{I}_{\sigma_1}^\vee \otimes \dots \otimes \mathcal{I}_{\sigma_n}^\vee$$

is also invertible and its formation naturally commutes with any base change on S .

Thanks to this good behavior for base change and Lemma 14.2.1, our task now becomes showing openness for the set of points $s \in S$ for which $\tilde{\omega}_{X/S}|_{X_s} \simeq \tilde{\omega}_{X_s/k(s)}$ is ample (this isomorphism of $\mathcal{O}_{X_s}$ -modules is an instance of compatibility of the formation of $\tilde{\omega}_{X/S}$ with respect to base change along $\mathrm{Spec}(k(s)) \rightarrow S$). This openness is an instance of the astounding [EGA, IV₃, 9.9.6] (amazing due to having no flatness hypotheses, which rules out accessing base change theorems for cohomology): for any proper map $f : X \rightarrow S$ of finite presentation and invertible $\mathcal{L}$ on X , the subset

$$\Omega = \{s \in S \mid \mathcal{L}|_{X_s} \text{ is ample on } X_s\}$$

is open in S and for all affine open $\mathrm{Spec} A \subset \Omega$ the restriction $\mathcal{L}_A := \mathcal{L}|_{X_A}$ to the A -proper X_A is A -ample (i.e. for some closed immersion $X_A \xrightarrow{j} \mathbf{P}_A^N$ over A into some projective space, $\mathcal{L}_A^{\otimes m} \simeq j^* \mathcal{O}_{\mathbf{P}_A^N}(1)$ for some $m > 0$). $\square$

At the end of the preceding proof we mentioned the strong “relative ampleness” property: fiberwise ampleness (in the proper and finitely presented case) implies usual ampleness over affine opens in the base. That isn't logically needed in the argument above, but this property (very striking due to its validity without any flatness hypotheses) will be absolutely crucial for effective descent of stable n -pointed genus- g curves, a key ingredient in having a good theory of moduli “spaces” for such structures.

Lecture 15

Moduli of stable curves

§15.1 Universal rigidified family over a Hilbert scheme

Recall the situation where we left off:

$$\begin{array}{ccccccc}
 Z = \cup \sigma_i(Y) & \hookrightarrow & X & \longleftarrow & X_U & \longleftarrow & Z_U = \coprod_{i=1}^n U \\
 & \searrow & \downarrow f & & \downarrow & & \\
 & & Y & \longleftarrow & U & &
 \end{array}$$

where $X_U \rightarrow U$ has smooth geometrically connected fibers of genus $g \geq 0$, and $n \geq 3$. We will eventually “alter” $(X, Z) \rightarrow Y$ to a stable marked curve (over Y) then alter Y to be smooth (by induction). This requires using the moduli stack $\overline{\mathcal{M}}_{g,n}$ which we shall see is smooth and proper over k . There exists a universal stable marked curve $\mathcal{C}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$ such that the following diagram commutes

$$\begin{array}{ccc}
 X_U & \longrightarrow & \mathcal{C}_{g,n} \\
 \downarrow & & \downarrow \sigma \\
 Y \supset U & \longrightarrow & \overline{\mathcal{M}}_{g,n}
 \end{array}$$

(including that the universal markings pull back to the ordered decomposition of Z as a union of sections).

Beware that the set-valued functor

$$S \rightsquigarrow \overline{M}_{g,n}(S) = \{\text{stable } n\text{-pointed genus-}g \text{ curves over } S\} / \simeq$$

is generally *not* representable when $g > 0$ because for each positive g some objects being classified admit non-trivial automorphisms:

EXAMPLE 15.1.1. For an example with $(g, n) = (1, 1)$, let E be an elliptic curve over a nonzero $\mathbf{Z}[1/2]$ -algebra R (such as a field k with $\text{char}(k) \neq 2$). Consider the “constant” elliptic curve $E \times \mathbf{G}_m \rightarrow \mathbf{G}_m$ with marking at the identity section $\sigma_1 : t \mapsto (0, t)$. The automorphism $(x, t) \mapsto (-x, -t)$ of $E \times \mathbf{G}_m$ over the automorphism $t \mapsto -t$ of $\mathbf{G}_m$ amount to free actions of $(\mathbf{Z}/(2))_R = \mu_{2,R}$ on each, and passing to the quotients then gives a stable 1-pointed genus-1 curve $C \rightarrow \mathbf{G}_m = S$ (the base is $\mathbf{G}_m/\mu_2 \simeq \mathbf{G}_m$ via $t \mapsto t^2$).

The curve $C \rightarrow S$ pulls back to the “constant” family $E \times \mathbf{G}_m \rightarrow \mathbf{G}_m$ over the degree-2 finite étale cover $t^2 : \mathbf{G}_m \rightarrow S$ but it is not itself isomorphic to $E \times S \rightarrow S$, so $\overline{M}_{1,1}$ is *not* a sheaf for the étale topology (the map $\overline{M}_{1,1}(S) \rightarrow \overline{M}_{1,1}(\mathbf{G}_m)$ is not even injective) and hence

cannot be represented by a scheme (nor can it be an algebraic space).

EXAMPLE 15.1.2. The presence of non-trivial automorphisms is unavoidable with reducible curves over algebraically closed fields when $g \geq 1$, no matter how big we make n . For example, let C be a smooth connected proper curve of genus $g - 1$ over an algebraically closed field k , and equip it with $n > 0$ marked points. Let X be obtained from gluing an elliptic curve E to C by identifying $0 \in E(k)$ with some $P \in C(k)$ distinct from the marked points. This X is a stable n -pointed genus- g curve, and inversion on E is a non-trivial automorphism of the stable marked curve X .

§15.2 The moduli stack $\overline{\mathcal{M}}_{g,n}$

To remove nontrivial automorphisms, one idea is to equip data with “enough” extra structure. If attached to each object $f : \xi \rightarrow S$ is a “well-behaved” fiberwise ample $\mathcal{L}_\xi$ (e.g., with $f_*(\mathcal{L}_\xi^{\otimes 100})$ a vector bundle of some rank N determined by known discrete data; here 100 is stand-in for some large integer) then we can try

$$\begin{array}{ccccc} \xi & \hookrightarrow & \mathbf{P}(f_*(\mathcal{L}_\xi^{\otimes 100})) & \xrightarrow[\alpha]{\sim} & \mathbf{P}_S^{N-1} \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

with α some isomorphism attained after possibly shrinking S .

Appendices C.2 and C.3 give us the following: For any $(X, \underline{\sigma})$ over S , we define $\mathcal{L}_{X/S, \underline{\sigma}} = (\omega_{X/S}(\sum \sigma_i))^{\otimes 4}$ (sometimes more briefly denoted $\mathcal{L}_{X/S}$ though it always depends on $\underline{\sigma}$). This is fiberwise ample and satisfies:

- 1) $\mathcal{L}_{X/S}$ is canonical with respect to isomorphisms in $(X, \underline{\sigma})$ and its formation naturally commutes with any base change on S .
- 2) (using cohomology and base change) $f_*\mathcal{L}_{X/S}$ is a vector bundle on S whose formation naturally commutes with any base change on S , and its rank is $N = N(g, n, 4) = 4(2g - 2 + n) + (1 - g)$.
- 3) $\mathcal{L}_{X/S}$ is generated by global sections; i.e. $f^*(f_*\mathcal{L}_{X/S}) \rightarrow \mathcal{L}_{X/S}$ is *surjective*, defining a *closed immersion*

$$\begin{array}{ccc} X & \hookrightarrow & \mathbf{P}(f_*\mathcal{L}_{X/S}) \\ & \searrow & \downarrow \\ & & S \end{array}$$

where $\mathbf{P}(f_*\mathcal{L}_{X/S})$ is isomorphic to $\mathbf{P}^{N-1}$ Zariski-locally on S and $X \rightarrow S$ has fibral Hilbert polynomial with respect to $\mathcal{L}_{X/S}$ given by $\Phi_{g,n}(t) = 4(2g - 2 + n)t + (1 - g) \in \mathbf{Z}[t]$.

For the (quasi-projective) Hilbert scheme $H = \text{Hilb}_{\mathbf{P}_{\mathbf{Z}}^{N-1}/\mathbf{Z}}$ and the universal flat family with Hilbert polynomial Φ

$$\begin{array}{ccc} \mathfrak{Z} & \hookrightarrow & \mathbf{P}_H^{N-1} \\ & \searrow & \downarrow \\ & & H \end{array}$$

the n -fold fiber power $\mathfrak{Z}^n = \mathfrak{Z} \times_H \cdots \times_H \mathfrak{Z}$ represents the functor of flat families

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbf{P}_T^{N-1} \\ & \searrow & \downarrow \\ & & T \end{array}$$

equipped with an ordered n -tuple $\tau_1, \dots, \tau_n \in Y(T)$. From our discussion of openness conditions on the base last time, there is an open $\Omega \subset \mathfrak{Z}^n$ representing such $(Y, \underline{\tau})$ that are stable n -pointed genus- g curves (for a fixed choice of g and n satisfying $2g - 2 + n > 0$).

Beware that for the universal object

$$\begin{array}{ccc} (\mathcal{C}, \underline{\sigma}) & \xrightarrow{j} & \mathbf{P}_{\Omega}^{N-1} \\ & \searrow & \downarrow \\ & & \Omega \end{array}$$

there is no specified relation between $j^*\mathcal{O}(1)$ and $\mathcal{L}_{\mathcal{C}/\Omega, \underline{\sigma}}$, nor between j and the *canonical* closed immersion $\iota_{\mathcal{C}, \underline{\sigma}}$ of $\mathcal{C}$ into the Zariski $\mathbf{P}^{N-1}$ -bundle $\mathbf{P}(f_*(\mathcal{L}_{\mathcal{C}/\Omega, \underline{\sigma}}))$.

By imposing a closed condition on a PGL_N -torsor Isom-scheme over Ω , we can universally identify $j^*\mathcal{O}(1)$ with $\mathcal{L}_{\mathcal{C}/\Omega, \underline{\tau}}$ in such a way that j is identified with $\iota_{\mathcal{C}, \underline{\sigma}}$; this is explained as the main conclusion in Appendices C.2 and C.3). The upshot is that we can build a quasi-projective $\mathbf{Z}$ -scheme $\overline{Y}_{g,n}$ that represents the functor

$$S \rightsquigarrow \{(\mathcal{C}, \underline{\sigma}) \rightarrow S, \varphi : \mathbf{P}(f_*\mathcal{L}_{\mathcal{C}/S}) \simeq \mathbf{P}_S^{N-1}\}$$

(such data $((\mathcal{C}, \underline{\sigma}), \varphi)$ admit no nontrivial automorphisms). Morally this says that $\overline{Y}_{g,n}$ is a Zariski PGL_N -torsor over the “fibered category” $\overline{\mathcal{M}}_{g,n}$ defined by:

DEFINITION 15.2.1. For any scheme S , $\overline{\mathcal{M}}_{g,n}(S)$ denotes the category of pairs $(\mathcal{C}, \underline{\sigma})$ over S with isomorphisms as the only maps; i.e. this is a groupoid. We make $S \rightsquigarrow \overline{\mathcal{M}}_{g,n}(S)$ into a “presheaf” in groupoids (called a “fibered category”) via base change: for $S' \rightarrow S$ the associated base change functor

$$b_{S'/S} : \overline{\mathcal{M}}_{g,n}(S) \rightarrow \overline{\mathcal{M}}_{g,n}(S')$$

is equipped with a “nice” isomorphism of functors $b_{S''/S} \simeq b_{S''/S'} \circ b_{S'/S}$ (namely satisfying an evident associativity requirement with respect to composable maps $S''' \rightarrow S'' \rightarrow S' \rightarrow S$).

Since the *canonically* associated $\tilde{\omega}_{X/S}$ is relatively ample in the sense discussed via the miraculous [EGA, IV₃, 9.9.6] last time, fpqc descent theory (see [BLR90, §6.1]) ensures that for $S' \rightarrow S$ fpqc, the fully faithful functor

$$\overline{\mathcal{M}}_{g,n}(S) \rightarrow \left\{ (X', \theta) \mid X' \in \overline{\mathcal{M}}_{g,n}(S'), \theta : p_1^*(X') \xrightarrow{\sim} p_2^*(X') \text{ in } \overline{\mathcal{M}}_{g,n}(S' \times_S S') \text{ satisfying cocycle} \right\}$$

via $X \mapsto (X_{S'}, \theta_{\text{can}})$ is an equivalence (where the descent from S' to S is unique up to unique isomorphism). In other words, $\overline{\mathcal{M}}_{g,n}$ is a “stack in groupoids” for the fpqc and Zariski topologies. Moreover, the diagonal

$$\Delta : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n} \times_{\mathbf{Z}} \overline{\mathcal{M}}_{g,n} \quad (15.2.1)$$

(whose target is defined as the fibered category of *ordered pairs* $((X, \underline{\sigma}), (Y, \underline{\tau}))$ over any base scheme S) is “relatively representable” in schemes in the sense that the Isom-functor between any two objects $(X, \underline{\sigma})$ and $(Y, \underline{\tau})$ over a scheme S is represented by a locally finite type S -scheme (this representability follows from Grothendieck’s work on Hilbert schemes and Isom-schemes because such data is projective Zariski-locally on the base thanks to the relatively ample line bundles $\tilde{\omega}_{(X/S)}$ and $\tilde{\omega}_{(Y/S)}$).

Thus, $\overline{\mathcal{M}}_{g,n}$ is a quasi-compact Artin stack locally of finite type over $\mathbf{Z}$ due to the PGL_N -torsor scheme $\overline{Y}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$. The Artin stack $\overline{\mathcal{M}}_{g,n}$ is even of “finite presentation” over $\mathbf{Z}$, which is to say it is also quasi-separated: this says that the relatively representable diagonal (15.2.1) is represented in *quasi-compact* scheme morphisms: for objects $(X, \underline{\sigma})$ and $(X', \underline{\sigma}')$ over a scheme S , the Isom-functor $\text{Isom}((X, \underline{\sigma}), (X', \underline{\sigma}')) \rightarrow S$ is represented by an S -scheme whose structure map to S is *quasi-compact*.

In fact, one can prove more, as Deligne and Mumford knew at the time they wrote [DM69] (when the concept of Artin stack didn’t yet exist, forcing Deligne and Mumford to use more direct “étale slicing” arguments): $\overline{\mathcal{M}}_{g,n}$ admits an *étale* cover by a scheme (not just a smooth cover, such as $\overline{Y}_{g,n}$), which is to say it is a Deligne-Mumford stack. Here is the result:

THEOREM 15.2.2. The quasi-compact locally finite type Artin stack $\overline{\mathcal{M}}_{g,n}$ over $\mathbf{Z}$ is Deligne-Mumford and quasi-separated.

Proof. Stable marked curves are projective Zariski-locally over the base, so by Grothendieck’s existence results for Isom-schemes using his work on Hilbert schemes (via graphs inside a product) we know that the Isom-functor between the underlying curves is represented locally on the base by a disjoint union of at most countably many quasi-projective schemes. The condition of preserving the markings cuts out a closed subscheme in the “unmarked” Isom-scheme. Since the clopen locus in a Hilbert scheme corresponding to a fixed Hilbert polynomial is quasi-compact over the base, for

quasi-compactness over the base for the Isom-scheme for a pair of stable n -pointed genus- g curves it suffices to check that the graph

$$\Gamma_\varphi \hookrightarrow X \times X'$$

of an isomorphism $\varphi : (X, \underline{\sigma}) \simeq (X', \underline{\sigma}')$ has Hilbert polynomial with respect to

$$\mathcal{L} := p_1^*(\mathcal{L}_{X/S, \underline{\sigma}}) \otimes p_2^*(\mathcal{L}_{X'/S, \underline{\sigma}'})$$

determined just by g and n . But the factors X and X' have the same Hilbert polynomial $\Phi(t) := \Phi_{g,n}(t) = 4(2g - 2 + n)t + (1 - g)$ relative to their respective S -ample line bundles $\mathcal{L}_{X/S, \underline{\sigma}}$ and $\mathcal{L}_{X'/S, \underline{\sigma}'}$, so their product is readily seen to have Hilbert polynomial $\Phi(2t)$ with respect to $\mathcal{L}$.

We have shown quasi-separatedness, so $\overline{\mathcal{M}}_{g,n}$ is a finitely presented Artin stack over $\mathbf{Z}$. It is a general fact [LMB, Thm. 8.1] (or for more generality without finiteness conditions see [SP, Tags 04YW(1), 06N3, 050N(1)]) that a finitely presented Artin stack over a scheme is Deligne-Mumford if its finite type diagonal is “unramified”, which amounts to the (algebraic space) Isom-functor between any pair of objects over a base having vanishing relative Ω^1 . By Nakayama’s Lemma such vanishing is sufficient to check over geometric points of the base, where the Isom-space is either empty of an Aut-scheme. In other words, we need the Aut-scheme of objects over algebraically closed field to be étale. For the Artin stack $\overline{\mathcal{M}}_{g,n}$, this étaleness is part of Lemma 14.1.1. $\square$

Lecture 16

Some generalities on stacks

§16.1 What is an Artin or Deligne-Mumford stack?

DEFINITION 16.1.1. A **fibered groupoid** (or “presheaf in groupoids”) $\mathcal{X}$ over the category of schemes is an assignment of a groupoid $\mathcal{X}(T)$ to each scheme T along with a functor (“base change”) $f^* : \mathcal{X}(T) \rightarrow \mathcal{X}(T')$ for each $f : T' \rightarrow T$ (we often write $x_{T'}$ rather than $f^*(x)$ for objects $x \in \mathcal{X}(T)$) and isomorphisms

$$b_{g,f} : g^* \circ f^* \simeq (f \circ g)^*$$

for $T'' \xrightarrow{g} T' \xrightarrow{f} T$ satisfying the evidence associativity requirement that the diagram of natural isomorphisms

$$\begin{array}{ccccc} h^* \circ (g^* \circ f^*) & \xrightarrow{\simeq} & h^* \circ (f \circ g)^* & \xrightarrow{\simeq} & ((f \circ g) \circ h)^* \\ \simeq \downarrow & & & & \parallel \\ (h^* \circ g^*) \circ f^* & \xrightarrow{\simeq} & (g \circ h)^* \circ f^* & \xrightarrow{\simeq} & (f \circ (g \circ h))^* \end{array}$$

commutes for any triple of composable maps $T''' \xrightarrow{h} T'' \xrightarrow{g} T' \xrightarrow{f} T$.

We call this a **stack** (fibered in groupoids) for the fppf topology on the category of schemes if it satisfies the *descent condition* that (i) for any faithfully flat and locally finitely presented map $S' \rightarrow S$ the natural transformation

$$\mathcal{X}(S) \rightarrow \{(\xi', \theta) \mid \xi' \in \mathcal{X}(S'), \theta : p_1^*(\xi') \xrightarrow{\sim} p_2^*(\xi') \text{ in } \mathcal{X}(S' \times_S S') \text{ satisfying the cocycle condition in } \mathcal{X}(S' \times_S S' \times_S S')\}$$

given by $\xi \mapsto (\xi_S, \theta_{\text{can}})$ is an equivalence (in effect, descent is effective up to unique isomorphism), and (ii) $\mathcal{X}(\coprod S_i) \rightarrow \prod \mathcal{X}(S_i)$ is an equivalence for any set $\{S_i\}$ of schemes (this is a stack condition for the Zariski topology when combined with the special case $S' = \coprod U_i$ in (i) for an open covering $\{U_i\}$ of any S).

The preceding is abstract categorical nonsense, with no relation to geometry. To equip such an $\mathcal{X}$ with “geometric structure”, we need to impose some more requirements. First we discuss some preliminary notions. By a Yoneda-type argument, for any scheme X a natural transformation $\varphi : \text{Hom}(-, X) \rightarrow \mathcal{X}$ as fibered categories over Sch (i.e., a specification of natural transformations $\varphi_T : X(T) \rightarrow \mathcal{X}(T)$ natural in arbitrary T , where the set $X(T)$ is regarded as a discrete category) is given by $f \mapsto f^*(\xi)$ for an object $\xi \in \mathcal{X}(X)$. Such φ is also called a “1-morphism” $X \rightarrow \mathcal{X}$.

For natural transformations $f : \mathcal{Y} \rightarrow \mathcal{Z}$ and $f' : \mathcal{Y}' \rightarrow \mathcal{Z}$ as fibered categories (i.e., these are

“1-morphisms”) we define the fibered category $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}'$ by

$$(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}')(S) := \{(\xi, \xi', \phi) \mid \xi \in \mathcal{Y}(S), \xi' \in \mathcal{Y}'(S), \phi : f(\xi) \xrightarrow{\sim} f'(\xi') \text{ in } \mathcal{Z}(S)\}.$$

For the link to algebraic geometry, we also need the concept of algebraic space. We’ll discuss what an algebraic space is soon, but for now you can think about it as a contravariant set-valued functor on Sch that is “representable up to étale quotients”. It is a mild but brilliant generalization of the notion of representable functor. By systematic use of étale descent, one can develop the entirety of algebraic geometry for such “nearly representable” functors on Sch . The true miracle, as discovered by Artin, is that algebraic spaces are the right broader context for powerful representability results far beyond what Grothendieck could prove for schemes via projective methods.

Coming back to our stack $\mathcal{X}$, to impose geometric structure we first require that the diagonal $\mathcal{X} \xrightarrow{\Delta_{\mathcal{X}}} \mathcal{X} \times \mathcal{X}$ is *relatively representable in algebraic spaces* in the sense that for the pullback square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times \mathcal{X} \\ \uparrow & & \uparrow (x,y) \\ \underline{\text{Isom}}_T(x, y) & \longrightarrow & T \end{array}$$

with any objects $x, y \in \mathcal{X}(T)$, the associated Isom-functor on T -schemes in the lower-left (its value on any $T' \rightarrow T$ is the set $\text{Isom}_{T'}(x_{T'}, y_{T'})$) is an algebraic space. This diagonal condition can be equivalently formulated as saying that for any schemes T, T' with maps $T \rightarrow \mathcal{X}$ and $T' \rightarrow \mathcal{X}$, the fiber product $T \times_{\mathcal{X}} T'$ is an algebraic space.

REMARK 16.1.2. For what we’ll need with $\mathcal{X} = \overline{\mathcal{M}}_{g,n}$, the Isom-functors are even represented by schemes (separated of finite type over the base) due to Grothendieck’s work on Hilbert schemes in the projective case. Such representability of Isom-functors by schemes tends to hold for moduli problems involving proper objects in the presence of enough flatness conditions and sufficiently functorial fiberwise-ample line bundle (as the latter tends to relate the situation to projective objects).

But there are entirely natural moduli problems for schemes where there is no ample line bundle encoded and consequently no relation to projective methods. For example, consider a proper finitely presented map of schemes $X \rightarrow S$ and define the contravariant set-valued Hilbert functor $\underline{\text{Hilb}}_{X/S}$ on the category of S -schemes by setting $\underline{\text{Hilb}}_{X/S}(T)$ to be the *set* of T -flat finitely presented closed subschemes $Z \subset X_T$. This is a sheaf for the fppf topology (even fpqc topology) by effective descent for closed subschemes (no need for an ample line bundle!).

It is a theorem for Artin [Art69, Cor. 6.2] (also see [Art74, Appendix] for corrections) from his early work on algebraic spaces that $\underline{\text{Hilb}}_{X/S}$ is an algebraic space over S (separated and locally of finite presentation over S); he even proved this when X is an algebraic space rather

than a scheme. Thus, by using graph arguments just as one does to build Isom-schemes from Hilbert schemes in the projective setting, it follows that the Isom-functor $\text{Isom}_S(X, Y)$ between a pair of proper flat and finitely presented schemes $X, Y \rightrightarrows S$ is an algebraic space but it has no reason to be a scheme.

This becomes relevant if one wants to organize all “genus- g ” curves into a stack. More specifically, for a scheme S let’s define a *genus- g curve* over S to be a proper flat and finitely presented algebraic space $f : C \rightarrow S$ with fibers of pure dimension 1 such that $\mathcal{O}_S \simeq f_*(\mathcal{O}_C)$ universally and $R^1 f_*(\mathcal{O}_C)$ is a vector bundle of rank g . An example with $g = 1$ beyond the scheme setting is [here](#).

The fibered groupoid of genus- g curves is a stack for the fppf topology – in other words, it satisfies effective descent for the fppf topology on the base – precisely because Artin proved [Art74, Cor. 6.3] fppf (and even just étale) descent is *effective* for finite presented algebraic spaces (this is generally not true for schemes, even if proper but without an ample line bundle carried along). It is also an Artin stack, as shown [here](#). The Isom-functors for this stack, as in the study of relative representability for its diagonal, are Isom-functors between such abstract genus- g curves. But in the absence of any smoothness conditions on the curves or any sections or allowing $g \leq 1$, even if the two curves are schemes there is no associated ample line bundle in sight and so no reason for the Isom-functor between them to be represented by a scheme. In general the best we can say is that such Isom-functors are algebraic spaces. This applies in particular with $g = 1$ and we impose smoothness on the curves (but no section!).

DEFINITION 16.1.3. An **Artin stack** is a stack $\mathcal{X}$ for the fppf topology over the category of schemes such that:

1. the diagonal morphism $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is relatively representable in algebraic spaces (so $T \times_{\mathcal{X}} T'$ is an algebraic space for any schemes T, T' and maps $T, T' \rightrightarrows \mathcal{X}$),
2. there exists a scheme X and a map $f : X \rightarrow \mathcal{X}$ that is smooth and surjective (in the sense that for any $S \rightarrow \mathcal{X}$ for a scheme S , the base change $X \times_{\mathcal{X}} S \rightarrow S$ of f is a smooth surjection between algebraic spaces).

If we instead impose there exists an *étale* surjection $X \rightarrow \mathcal{X}$ then we say $\mathcal{X}$ is a *Deligne-Mumford* stack (or DM stack); these were the type introduced in [DM69].

EXAMPLE 16.1.4. The stack $\mathcal{X} = \overline{\mathcal{M}}_{g,n}$ is an Artin stack with the smooth surjection $\overline{Y}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ that is even affine (a PGL_N -torsor for $N = 4(2g - 2 + n) + (1 - g)$). Morally, $\overline{\mathcal{M}}_{g,n}$ is the “stack quotient” of $\overline{Y}_{g,n}$ by the *non-free* action of PGL_N .

By Theorem 15.2.2 the Artin stack $\overline{\mathcal{M}}_{g,n}$ is a DM-stack. Although it has the concrete smooth scheme cover $\overline{Y}_{g,n}$, an étale scheme cover of $\overline{\mathcal{M}}_{g,n}$ is very hard to “write down” (it

comes from Artin approximation).

Let's now answer: what is an algebraic space? This is the notion obtained by carrying out the following special case of the preceding definitions: require each $\mathcal{X}(T)$ is a set (as a discrete category), the Isom-functors are represented by schemes, and there exists an étale surjection $X \rightarrow \mathcal{X}$ for X a scheme (i.e., $X \times_{\mathcal{X}} S$ is an étale S -scheme for every $S \rightarrow \mathcal{X}$).

In more explicit terms, algebraic spaces turn out to be exactly the set-valued functors U/R on Sch obtained by étale sheafification of $T \mapsto U(T)/R(T)$ where $R \rightarrow U \times U$ is an “étale equivalence relation”: a monomorphism of schemes such that (i) it is an equivalence relation on T -valued points for all T , (ii) the two maps $R \rightrightarrows U$ are étale.

The substantive fact is that one really can adapt the entire edifice of algebraic geometry to such gadgets. Beware of a crucial point: algebraic spaces are *not* ringed spaces (except when they are schemes). They do have an associated topological space, as is useful for defining notions such as connectedness, quasi-compactness, and irreducible component, but they are just set-valued functors on Sch (close enough to representable ones via the étale topology that it is possible to “do geometry” with them).

Maps between algebraic spaces are defined to be simply natural transformations of functors. With experience, one can use choices of étale scheme covers to make calculations for algebraic spaces in the same style by which one uses coordinate balls to make calculations for manifolds (e.g., a morphism between two algebraic spaces can also be described in terms of a special type of map between choices of étale scheme covers).

§16.2 Properties of Artin stacks over a base scheme

We will often work in terms of an Artin stack $\mathcal{X}$ over a scheme S , so we need to discuss two equivalent ways to think about this. What is a “map” $\varphi : \mathcal{X} \rightarrow S$? This means (by definition) a specification of functors $\mathcal{X}(T) \rightarrow \text{Hom}(T, S)$ natural in T . We shall now see that such data can be equivalently encoded by running the previous definition of Artin stack using the category Sch_S of S -schemes in place of Sch everywhere (and taking fiber products over S); let's temporarily call the latter notion an *Artin stack over S* .

To pass between these two perspectives, proceed as follows. Given φ as above, we get a fibered category $\mathcal{X}'$ on Sch_S by setting $\mathcal{X}'(T \xrightarrow{f} S)$ to be the fiber of $\varphi_T : \mathcal{X}(T) \rightarrow \text{Hom}(T, S)$ over f . This $\mathcal{X}'$ is checked to be an Artin stack over S .

To go the other way, if $\mathcal{X}'$ is an Artin stack over S then we define the fibered category $\mathcal{X}$ on Sch by setting $\mathcal{X}(T)$ to be the groupoid of pairs (f, ξ) where $f : T \rightarrow S$ is a morphism and $\xi \in \mathcal{X}'(T \xrightarrow{f} S)$. This is checked to be an Artin stack in the absolute sense defined earlier, and the map $\mathcal{X} \rightarrow S$ is defined by $(f, \xi) \mapsto f$.

These two procedures are readily seen to be inverses of each others, so we will now interchangeably speak of an Artin stack over S in either sense: an absolute notion equipped with a map to S or the

more relative notion as a stack fibered in groupoids over the base category of S -schemes (rather than all schemes).

Before we define an array of structural properties for Artin stacks over S , we need one last concept: the *associated topological space* $|\mathcal{X}|$ for an Artin stack is a certain quotient of $|X|$ for any choice of smooth scheme cover $X \rightarrow \mathcal{X}$ (we omit the definition, but it is naturally independent of the choice of such cover ultimately because smooth maps are open).

It turns out that open subsets of $|\mathcal{X}|$ can be related to certain open subsets of a choice of smooth scheme cover, and likewise for closed subsets. Beware that in general $|\mathcal{X}|$ can be very pathological (e.g., generic points not behaving as nicely as for schemes), but for us such pathologies won't arise.

DEFINITION 16.2.1. An Artin stack $\mathcal{X} \rightarrow S$ over a scheme S is:

1. *smooth* (resp. *locally finite type*, *locally finitely presented*) over S if for some (and then any) smooth scheme cover $X \rightarrow \mathcal{X}$, the composite map $X \rightarrow S$ is smooth (resp. locally finite type, locally finitely presented),
2. *quasi-separated* if $\mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is represented by quasi-compact and quasi-separated maps (in algebraic spaces),
3. *quasi-compact* if for some (and then any) affine open cover $\{U_i\}$ of S , each $\mathcal{X}_{U_i}$ admits a quasi-compact smooth scheme cover $X_i \rightarrow \mathcal{X}_{U_i}$,
4. *finitely presented* if it is locally finitely presented, quasi-separated, and quasi-compact over S ,
5. *separated* if $\mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is proper,
6. *finite type* if it is quasi-compact and locally finite type,
7. *proper* if it is separated, finite type, and universally closed (i.e. for all $T \rightarrow S$, $|\mathcal{X}_T| \rightarrow |T|$ is closed).

REMARK 16.2.2. In condition (1), the choice of smooth scheme cover doesn't matter. This is because descent theory ensures for a scheme diagram $Z' \xrightarrow{\text{sm}} Z \rightarrow S$, Z' is smooth (resp. locally finite type, resp. locally of finite presentation) over S if and only if Z is.

In condition (2), the quasi-separatedness condition on the diagonal is automatic when $\mathcal{X}$ is a scheme (or algebraic space); i.e., when there is no “stackiness”. This is because in such cases $\Delta_{\mathcal{X}/S}$ is then a *monomorphism* between set-valued functors, and a monomorphism f is always separated since its own diagonal Δ_f is even an *isomorphism* (hence a closed immersion!) by the very meaning of “monomorphism”.

EXAMPLE 16.2.3. The DM-stack $\overline{\mathcal{M}}_{g,n}$ is finite type over $\mathbf{Z}$ since the smooth scheme cover $\overline{Y}_{g,n}$ is finite type (i.e., quasi-compact and locally finite type). It is likewise locally finitely presented, and we claim it is also quasi-separated (i.e., its diagonal is quasi-compact and quasi-separated), hence finitely presented over $\mathbf{Z}$. (This is part of the statement of Theorem 15.2.2, but at the time the definition of such terminology had not been given though the essential content of its proof was carried out there, as we shall now review.)

The quasi-compactness and quasi-separatedness of $\Delta_{\overline{\mathcal{M}}_{g,n}/\mathbf{Z}}$ amounts to Isom-schemes between two stable n -pointed genus- g curves being quasi-compact and quasi-separated over the base. As discussed in the proof of Theorem 15.2.2, such Isom-schemes are even quasi-compact and *separated* over the base due to their construction from a quasi-projective “bounded” part of a Hilbert scheme locally on the base.

Here are some additional useful facts. First we have an infinitesimal smoothness criterion, deduced from the scheme case using a smooth scheme cover:

THEOREM 16.2.4. If S is noetherian and $\mathcal{X} \rightarrow S$ is locally finitely presented, its smoothness is equivalent to an infinitesimal lifting condition: for $\mathrm{Spec}(A) \rightarrow S$ with artin local A having algebraically closed residue field and a square-zero ideal $I \subset A$, every object in $\mathcal{X}(A/I)$ lifts to $\mathcal{X}(A)$.

Note the preceding result does *not* involve also lifting isomorphisms (and Isom-functors may not be smooth, as already happens when deforming a supersingular elliptic curve to a non-CM elliptic curve in characteristic 0).

A proof of Chow’s Lemma for Deligne–Mumford stacks was not yet documented in the published literature at the time of [deJ96], though it had been stated long before as [DM69, Thm. 4.12]. Here is the statement:

THEOREM 16.2.5 (Chow’s Lemma for DM stacks). Let $\mathcal{Z}$ be a separated Deligne–Mumford stack of finite presentation over a noetherian scheme S . There exists a surjective *proper* morphism $Z \rightarrow \mathcal{Z}$ from a quasi-projective S -scheme Z such that the map is étale on a dense open subscheme of Z .

This is [LMB, Cor. 16.6.1] (where Deligne–Mumford stacks are required to be quasi-separated by definition, so in their terminology “finite type” is the same as “finite presentation” for such a stack over a noetherian base). The version for Artin stacks has to drop the “generically étale” condition, and was proved later by Gabber [Ols05, Thm. 1.1]; it is technically more difficult, and we do not need it.

THEOREM 16.2.6 (Valuative criterion for properness). For noetherian S , a finitely presented Artin stack $\mathcal{X}$ over S is proper if and only if for every dvr R over S with fraction field K , the following two conditions hold:

1. (uniqueness) $\mathcal{X}(R) \rightarrow \mathcal{X}(K)$ is fully faithful (i.e., $\text{Isom}_R(x, y) \rightarrow \text{Isom}_K(x_K, y_K)$ is bijective for objects $x, y \in \mathcal{X}(R)$),
2. (existence) for every $\xi \in \mathcal{X}(K)$ there exists a finite extension K'/K and a (discrete) valuation lifting that on K such that for the associated dvr R' the object $\xi_{K'}$ extends to $\mathcal{X}(R')$.

REMARK 16.2.7. 1. For $\mathcal{X}$ a scheme, it is a pleasant exercise to check this is equivalent to the usual valuative criterion for properness.

2. As for schemes, it suffices to check the valuative criterion for properness using only complete dvr's R with algebraically closed residue field. This is proved with the aid of Chow's Lemma for stacks (Theorem 16.2.5).

Our goal next time is to show that $\overline{\mathcal{M}}_{g,n} \rightarrow \text{Spec}(\mathbf{Z})$ is smooth and proper. This will be proved by induction on n with g fixed in three cases:

1. $g \geq 2$ and $n \geq 0$,
2. $g = 1$ and $n \geq 1$,
3. $g = 0$ and $n \geq 3$.

First we will treat the base cases of these three induction arguments. For $g \geq 2$, $\overline{\mathcal{M}}_{g,0}$ was shown to be smooth and proper over $\mathbf{Z}$ in [DM69] via deformation theory for smoothness and the valuative criterion for properness. For $g = 1$, $\overline{\mathcal{M}}_{1,1}$ was studied via deformation theory and the valuative criterion for properness in [DR73]. We'll say more about these next time.

For $g = 0$, we claim $\overline{\mathcal{M}}_{0,3} = \text{Spec } \mathbf{Z}$ (so everything is clear for it): this says that for every scheme S there is one object in $\overline{\mathcal{M}}_{0,3}(S)$ up to unique isomorphism. Indeed, it is $(\mathbf{P}_S^1, \{0, 1, \infty\})$ (which is readily seen to have no nontrivial S -automorphism). To prove this, the rigidity (i.e., absence of nontrivial automorphisms) over every base scheme reduces the verification of uniqueness to working Zariski-locally on the base. A 3-pointed stable marked curve of genus 0 is smooth (as we may see over algebraically closed fields), so it is classical that locally on the base with the 1st section it is identified to $(\mathbf{P}_S^1, \infty)$. The other two sections are everywhere disjoint in $\mathbf{A}_S^1$, so we can carry one of the other sections to 0 (locally on S) via an additive change of coordinate, and finally carry the final section to 1 via a multiplicative unit change of coordinate.

Lecture 17

Smoothness and properness of $\overline{\mathcal{M}}_{g,n}$

§17.1 Base of induction

We shall prove that $\overline{\mathcal{M}}_{g,n}$ is smooth and proper over $\mathbf{Z}$ via induction on n with g fixed, so the base cases are as follows: $g \geq 2$ with $n = 0$, $g = 1$ with $n = 1$, and $g = 0$ with $n = 3$. We saw $\overline{\mathcal{M}}_{0,3} = \mathrm{Spec}(\mathbf{Z})$ at the end of last time, so all is clear for this. In what follows, for ease of notation we'll write $\overline{\mathcal{M}}$ for $\overline{\mathcal{M}}_{g,0}$ (the base case for $g \geq 2$) or $\overline{\mathcal{M}}_{1,1}$ (the base case for $g = 1$).

We write $\mathcal{M} \subset \overline{\mathcal{M}}$ for the open substack corresponding to smooth proper genus- g curves (with geometrically connected fibers): this is the open locus of smooth fibers for the universal stable curve $\mathcal{X} \rightarrow \overline{\mathcal{M}}$. It corresponds to the open locus in any smooth scheme covering $X \rightarrow \overline{\mathcal{M}}$ over which the proper semistable pullback curve $f : C = \mathcal{X} \times_{\overline{\mathcal{M}}} X \rightarrow X$ has smooth fibers (this is open in X because (i) its complement is $f(C^{\mathrm{sing}})$ where $C^{\mathrm{sing}} \subset C$ is the closed subset complementary to the open X -smooth locus C^{sm} , and (ii) f is proper)..

To get started, we first discuss (with references) the smoothness of $\overline{\mathcal{M}}$ over $\mathbf{Z}$ for $g \geq 1$, then separatedness over $\mathbf{Z}$, and finally properness. We proceed in this order because the valuative criterion will be verified in a refined formulation that requires some structural knowledge about $\mathcal{M} \subset \overline{\mathcal{M}}$ emerging from the study of $\mathbf{Z}$ -smoothness of $\overline{\mathcal{M}}$. The refined version of the valuative criterion will rest on Chow's Lemma for stacks (Theorem 16.2.5) that requires a separatedness ingredient; this is why we have to treat separatedness before properness for $\overline{\mathcal{M}}$.

To prove smoothness over $\mathbf{Z}$ of $\overline{\mathcal{M}}$, we will use the infinitesimal lifting criterion from Theorem 16.2.4: we need to show $\overline{\mathcal{M}}(A) \rightarrow \overline{\mathcal{M}}(A/I)$ is essentially surjective for all Artin local A with algebraically closed residue field such that $I \subset A$ with $I^2 = 0$ (it even suffices to consider I for which $\mathfrak{m}I = 0$, by the same length-induction argument as for schemes). For this infinitesimal lifting property, see [DM69, §1] for $g \geq 2$ and see [DR73, Ch. II, Prop. 2.7, Ch. III, Thm. 1.2] for $g = 1$.

This deformation-theoretic work yields an explicit description of formal deformation rings at geometric points and the completed local ring for the algebraized universal curve over it at singularities in the special fiber. By inspecting these descriptions, one can deduce that the “boundary” of $\mathcal{M}$ in $\overline{\mathcal{M}}$ is Cartier in a strong sense: there exists an invertible ideal $\mathcal{I}_\infty \subset \mathcal{O}_{\overline{\mathcal{M}}}$ (here $\mathcal{O}_{\overline{\mathcal{M}}}(T \rightarrow \overline{\mathcal{M}}) := \mathcal{O}_T(T)$) such that $\mathcal{M} = \overline{\mathcal{M}} - Z(\mathcal{I}_\infty)$ as open substacks of $\overline{\mathcal{M}}$ with $Z(\mathcal{I}_\infty)$ even $\mathbf{Z}$ -flat.

(In effect, at the level of an étale or smooth scheme cover of $\overline{\mathcal{M}}$, the locus of singular fibers is cut out locally by a non-zero-divisor which remains such on all fibers over $\mathrm{Spec}(\mathbf{Z})$. The role of flatness here is similar to how flatness in the theory of Hilbert schemes ensures the ideal sheaf of the closed subscheme remains an *ideal sheaf*, not just an abstract quasi-coherent sheaf, after any base change.)

This is important because the complement of a Cartier divisor is always a *dense* open (i.e., for noetherian A , the open subset $\mathrm{Spec}(A_f) \subset \mathrm{Spec}(A)$ contains all generic points when f isn't a zero-divisor). Hence, the open substack $\mathcal{M} \subset \overline{\mathcal{M}}$ is dense. This is not obvious from the definitions: it says that singular objects over an algebraically closed field admit a smooth deformation. Such density will be used below to deploy a refined version of the valuative criterion.

As a bonus that we don't need here but is useful in other contexts, the flatness of $Z(\mathcal{I}_\infty)$ even gives it is “relatively schematically dense” over $\mathbf{Z}$ (i.e., dense and remains so after any base change, or equivalently is fiberwise-dense): see [EGA, IV₃, 11.10.8–10] for more on that notion in a general setting.

The ingredients are now in place to prove separatedness of $\overline{\mathcal{M}} \rightarrow \mathrm{Spec} \mathbf{Z}$. By definition, this means properness of $\Delta_{\overline{\mathcal{M}}/\mathbf{Z}}$ (which we already know is separated, since the Isom-schemes for stable curves are separated due to how they are made out of Hilbert schemes; they are even quasi-projective locally on the base). Consider the visibly cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}} & \xrightarrow{\Delta_{\overline{\mathcal{M}}/\mathbf{Z}}} & \overline{\mathcal{M}} \times \overline{\mathcal{M}} \\ \uparrow & & \uparrow \text{dense open} \\ \mathcal{M} & \xrightarrow{\Delta_{\mathcal{M}/\mathbf{Z}}} & \mathcal{M} \times \mathcal{M} \end{array}$$

We will now apply a refined version of the valuative criterion for separatedness:

PROPOSITION 17.1.1. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a separated and finitely presented map between Artin stacks with $\mathcal{Y}$ noetherian (i.e., $\mathcal{Y}$ is quasi-separated with a noetherian smooth scheme cover). Let $\mathcal{U} \subset \mathcal{X}$ be a dense open substack.

The map f is proper if and only if for all $\mathrm{Spec}(R) \rightarrow \mathcal{Y}$ with a complete dvr R having fraction field K and algebraically closed residue field, and every object $\xi \in \mathcal{U}(K)$, there exists a finite extension K'/K such that $\xi_{K'}$ extends to $\mathcal{X}(R')$ for the corresponding discrete valuation ring R' of K' .

Properties such as separatedness and finite presentation for such f whose target is also a stack means (by definition) that for some (and then any, by descent theory for schemes) smooth scheme cover $Y \rightarrow \mathcal{Y}$ the Artin stack $\mathcal{X} \times_{\mathcal{Y}} Y$ has this property over Y .

REMARK 17.1.2. Before we prove Proposition 17.1.1, let's explain the intuition by considering the case of a quasi-projective scheme $Y \rightarrow \mathrm{Spec}(A)$, say with projective closure $\overline{Y}$. Let $U \subset Y$ be dense open, so it contains all generic points. If Y is not A -proper then there exists $y \in \overline{Y} - Y$, yet $\overline{Y}$ as the closure of the dense open Y has all of its generic points in Y and so in U . Since y lies in some irreducible component, there is some generic point $\eta \in U$ with y in its closure. As a generization-specialization pair $\{\eta, y\}$ with $\eta \neq y$, there exists a discrete valuation ring $R \rightarrow \mathrm{Spec}(\mathcal{O}_{\overline{Y}, y}) \rightarrow Y$ with generic point going to η and special point going to y . We can

replace R with its completion, and even with a further local dvr extension with algebraically closed residue field.

This provides a K -valued point of Y that fails to extend to an R -valued point of Y (as it *does* extend to $\bar{Y}(R)$ by design, uniquely since $\bar{Y}$ is A -separated, yet this R -point specializes *outside* Y). In this way, we have built a violation of the valuative criterion from the failure of properness with K -point in U .

The preceding idea in the quasi-projective case can be adapted to more general settings with the aid of Chow's Lemma, as the proof below illustrates.

Proof. For necessity (which will hold even without the limitation to K -points being in $\mathcal{U}$), let $Y \rightarrow \mathcal{Y}$ be a smooth scheme cover. For any map $\mathrm{Spec}(R) \rightarrow \mathcal{Y}$, the pullback

$$P = Y \times_{\mathcal{Y}} \mathrm{Spec}(R)$$

is smooth surjective over $\mathrm{Spec}(R)$ and hence admits an R -point (as R is complete with algebraically closed residue field). Thus, by passing to $\mathcal{X} \times_{\mathcal{Y}} Y \rightarrow Y$ we can reduce necessity to the known case when the target is a noetherian scheme.

Sufficiency is the more interesting (and useful!) direction. We assume the more limited valuative criterion (with K -points in $\mathcal{U}$) and seek a contradiction if $\mathcal{X} \rightarrow \mathcal{Y}$ is *not* proper.

Let $h : Y \rightarrow \mathcal{Y}$ be a smooth surjection with Y noetherian. By flatness of h , the open substack

$$\mathcal{U}' := \mathcal{U} \times_{\mathcal{Y}} Y \subset \mathcal{X} \times_{\mathcal{Y}} Y =: \mathcal{X}'$$

is also dense. Likewise, by descent theory (since h is fppf), non-properness of $\mathcal{X} \rightarrow \mathcal{Y}$ implies non-properness of $f' : \mathcal{X}' \rightarrow Y$. But by design of $\mathcal{U}' \subset \mathcal{X}'$ the map f' satisfies the limited valuative criterion too (check!). Hence, for the purpose of reaching a contradiction we may and do now assume $\mathcal{Y}$ is a noetherian scheme, which we'll denote as Y .

Now we bring in Chow's Lemma: since $\mathcal{X} \rightarrow Y$ is separated of finite type with Y a noetherian scheme, there is a proper surjection $X \rightarrow \mathcal{X}$ from a scheme, so $X \rightarrow Y$ is a separated scheme map of finite type. Using Chow's Lemma for schemes [EGA, II, §5.6] applied to $X \rightarrow Y$, we can compose back to a further proper surjection onto X to arrange that X is even quasi-projective over Y , say X is open in a projective (hence proper) Y -scheme $\bar{X}$. Let $U \subset X$ be the open preimage of the dense open $\mathcal{U} \subset \mathcal{X}$. This might not be dense in X , but we can fix that as follows. By properness of $X \rightarrow \mathcal{X}$, the schematic closure $\bar{U} \subset X$ has closed image in $\mathcal{X}$. But that closed image contains the *dense open* $\mathcal{U} \subset \mathcal{X}$, so the the closed image exhausts $\mathcal{X}$ (at the level of associated topological spaces). In other words, the proper $\bar{U} \rightarrow \mathcal{X}$ is also surjective, so we can replace X with $\bar{U}$! In this way, we have arranged that U is also dense in X .

Since $\mathcal{X} \rightarrow Y$ is separated and $X \rightarrow \mathcal{X}$ is a proper surjection, arguing as in the case of schemes via universal closedness ensures that Y -properness of X is *equivalent to* (and in particular implies)

Y -properness of $\mathcal{X}$. But we are assuming $\mathcal{X}$ is *not* Y -proper, so likewise X is *not* Y -proper. But X is dense open in $\overline{X}$, so the dense open $U \subset X$ (!) is likewise dense open in the projective Y -scheme $\overline{X}$.

Since X is *not* Y -proper but is dense open in the Y -proper $\overline{X}$, there exists a point $x \in \overline{X} - X$. This point lies in some irreducible component of $\overline{X}$, and the generic point η of that lies in the dense open $U \subset \overline{X}$. Thus, $\{\eta, x\}$ is a pair of distinct points in $\overline{X}$ with x in the closure of η , so exactly as in Remark 17.1.2 there exists a complete dvr R with algebraically closed residue field and fraction field K and $\xi \in \overline{X}(R)$ such that $\xi_K \in U(K)$ and the special point of ξ lies over x . The composite map $\text{Spec}(R) \rightarrow \overline{X} \rightarrow Y$ then has corresponding K -point with a specified lift $\xi_K \in U(K) \subset X(K)$ which does *not* extend to $X(R)$ (since by the uniqueness aspect of Y -separatedness for $\overline{X}$ this would have to agree with $\xi \in \overline{X}(R)$, an absurdity since by design the special point of ξ lies over $x \in \overline{X} - X$).

We have used $\xi_K \in U(K)$ to exhibit a violation of the more limited valuative criterion for $X \rightarrow Y$ with the dense open $U = \mathcal{U} \times_{\mathcal{X}} X$. Observe that this violation is inherited after passing to any finite extension K'/K and the corresponding valuation ring $R' \subset K'$ over R .

Now consider the corresponding object $\tilde{u} \in \mathcal{U}(K)$ arising from ξ_K via $U \rightarrow \mathcal{U}$. We claim that for any finite extension K'/K and the corresponding valuation ring R' over R , $\tilde{u}_{K'}$ does *not* come from $\mathcal{X}(R')$ (so we will have reached the desired contradiction). Assume to the contrary, so we may rename K' as K to arrange that $\tilde{u}$ arises from some $\tilde{\xi} \in \mathcal{X}(R)$.

Let's now pull back the entire cartesian diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathcal{X} \end{array}$$

(with proper surjections vertically and open immersions horizontally) along $\tilde{\xi} : \text{Spec}(R) \rightarrow \mathcal{X}$. Since $\tilde{u} = \tilde{\xi}_K$ and $\tilde{u} \in \mathcal{U}(K)$ arises from $\xi_K \in U(K)$, the pullback diagram is cartesian of the form

$$\begin{array}{ccc} V & \longrightarrow & P \\ \downarrow & & \downarrow \\ \text{Spec}(K) & \longrightarrow & \text{Spec}(R) \end{array}$$

with proper surjections vertically. In this diagram $V(K)$ is non-empty since it has a K -point arising from ξ_K over $\tilde{u}$. By the usual valuative criterion applied to $P \rightarrow \text{Spec}(R)$, the point $\xi_K \in V(K)$ comes from a point in $P(R)$! By design of P , there exists an R -point of X which extends $\xi_K \in U(K)$, an impossibility since ξ_K comes from $\xi \in \overline{X}(R) - X(R)$ and this is the *only* R -point of $\overline{X}$ (let alone of X) which can extend ξ_K by the Y -separatedness of $\overline{X}$. $\square$

We now use the preceding refined valuative criterion to prove properness of the *separated* finite

type $\Delta := \Delta_{\mathcal{M}/\mathbf{Z}}$ with the aid of the dense open

$$\mathcal{M} \subset \overline{\mathcal{M}}.$$

It suffices to show that for any complete dvr R with fraction field K and algebraically closed residue field, and any $(x, y) \in (\overline{\mathcal{M}} \times \overline{\mathcal{M}})(R)$ (i.e., a pair of objects $x, y \in \overline{\mathcal{M}}(R)$) equipped with a K -point $\eta \in \mathcal{M}(K) \subset \overline{\mathcal{M}}(K)$ carried by Δ to (x_K, y_K) (i.e., η is an isomorphism $x_K \simeq y_K$), there exists $\xi \in \overline{\mathcal{M}}(R)$ extending η with $\Delta(\xi) = (x, y)$ (one has to be more precise about this “equality” among objects in a groupoid). In other words, we want to show that

$$\mathrm{Isom}_R(x, y) \rightarrow \mathrm{Isom}_K(x_K, y_K)$$

is surjective (it is injective by separatedness of Δ).

This is exactly the uniqueness part of the stable reduction theorem for *smooth* K -curves ($g \geq 1$, elliptic curves for $g = 1$): a stable R -model for a smooth curve over K (of genus $g > 0$, and with a section when $g = 1$) is unique up to unique isomorphism. This is proved in [DM69, §2] for $g \geq 2$ and [DR73, IV, Prop. 1.6(iv)] when $g = 1$. Its content amounts to there being at most one stable R -model up to isomorphism and bijectivity of $\mathrm{Aut}_R(\mathcal{C}, \underline{\sigma}) \rightarrow \mathrm{Aut}_K(\mathcal{C}_K, \underline{\sigma}_K)$. Note that an “ R -model” x of a given stable K -curve ξ is a stable R -curve *equipped with* an isomorphism $\varphi : x_K \simeq \xi$ of its K -fiber, so *changing* φ amounts to a different R -model. This is why “uniqueness up to unique isomorphism” for stable R -models entails exactly the bijection from an Isom-set over R to an Isom-set over K . Such strong uniqueness is intertwined with the theory of minimal regular proper models over a dvr for smooth geometrically connected proper curves of positive genus over the fraction field.

Finally, we can prove $\mathbf{Z}$ -properness of $\overline{\mathcal{M}}$, given that we now know it is *separated* (of finite presentation) over $\mathbf{Z}$. The refined valuative criterion with the dense open $\mathcal{M} \subset \overline{\mathcal{M}}$ over $\mathbf{Z}$ is exactly the existence part of the stable reduction theorem for *smooth* K -curves (this K -smoothness corresponds to the refined valuative criterion with $\mathrm{Spec} K$ landing in the open substack $\mathcal{M} \subset \overline{\mathcal{M}}$). Explicitly, it says that for any complete dvr R with algebraically closed residue field and fraction field K , and any smooth proper geometrically connected K -curve C with genus $g \geq 1$ that is equipped with a section when $g = 1$, there exists a finite (separable) extension K'/K such that $C_{K'}$ extends to a stable genus- g curve over the valuation ring R' (0-pointed when $g \geq 2$, and 1-pointed when $g = 1$). This is proved in [DM69, §2] for $g \geq 2$ and [DR73, IV, Prop. 1.6(i)-(iii)] for $g = 1$ (and $n = 1$).

§17.2 Stabilization and contraction

We now move on to the induction step. We aim to show $\overline{\mathcal{M}}_{g,n+1} \stackrel{?}{\simeq} \mathcal{Z}_{g,n}$ as fibered categories, where $\mathcal{Z}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the universal family (so this is a proper DM-stack over $\mathbf{Z}$). As a first step, we

describe the fiber categories $\mathcal{Z}_{g,n}(T)$ for schemes S . This will *not* be literally the same objects as in $\overline{\mathcal{M}}_{g,n+1}(T)$, but we'll see that the two types of objects over S are in natural correspondence with each other.

To get some intuition, recall from our discussion of Hilbert schemes that if $Z \rightarrow \text{Hilb}_{X/S}$ is the universal family (a flat closed subscheme of $X \times_S \text{Hilb}_{X/S}$) then the functor of points $Z(T)$ on an S -scheme T represents the set of *pair* (Y, τ) where $Y \in \text{Hilb}_{X/S}(T)$ is a T -flat finitely presented closed subscheme of X_T and $\tau \in Y(T)$. In other words, the functor of points of Z (over S) is the enhanced version of the base $\text{Hilb}_{X/S}$ in which we equip the flat family with a section.

For exactly the same reason, replacing $Z \rightarrow \text{Hilb}_{X/S}$ with $\mathcal{Z}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$, we have

$$\mathcal{Z}_{g,n}(S) = \{(\mathcal{C}, \sigma_1, \dots, \sigma_n; \sigma_{n+1})\}$$

where $(\mathcal{C}, \sigma_1, \dots, \sigma_n)$ is an n -pointed stable genus- g curve and $\sigma_{n+1} \in \mathcal{C}(S)$ is an arbitrary section. The key point is that $(\mathcal{C}, \sigma_1, \dots, \sigma_n, \sigma_{n+1})$ may *not* be stable as an $(n+1)$ -pointed genus- g curve over S . For example, σ_{n+1} may meet the non-smooth locus $\mathcal{C}^{\text{sing}} \rightarrow S$ or σ_{n+1} may meet σ_i for some $i \leq n$. For this reason, there isn't an "obvious" functor $\mathcal{Z}_{g,n}(S) \rightarrow \overline{\mathcal{M}}_{g,n+1}(S)$.

Going in the other direction, we ask whether there is a well-posed functor

$$\overline{\mathcal{M}}_{g,n+1}(S) \xrightarrow{?} \mathcal{Z}_{g,n}(S)$$

defined by "forget σ_{n+1} as a stable-curve marking, but retain it as a section:

$$(\mathcal{C}, \sigma_1, \dots, \sigma_{n+1}) \mapsto (\mathcal{C}, \sigma_1, \dots, \sigma_n; \sigma_{n+1}).$$

This doesn't work because $(\mathcal{C}, \sigma_1, \dots, \sigma_n)$ may *not* be stable as an n -pointed genus- g curve! (Even if that weren't an issue, this procedure could not be essentially surjective since it doesn't account for sections through the non-smooth locus in semistable genus- g curve.) There are three ways such failure of stability as an n -pointed curve can happen for a stable $(n+1)$ -pointed stable genus- g curve $(C, \sigma_1, \dots, \sigma_{n+1})$ over an algebraically closed field k (and it is an instructive exercise to check these are the *only* ways such failure can happen; keep in mind that $2g - 2 + n > 0$):

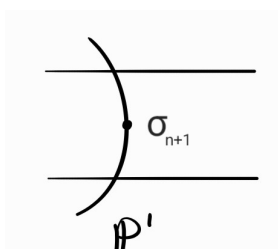


Figure 17.2.1: The point $\sigma_{n+1} \in C^{\text{sm}}$ lies on a copy of $\mathbf{P}^1$ that transversally meets two other irreducible components Y, Y' of C at one smooth point of each.

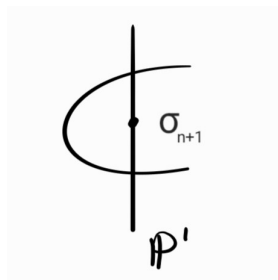


Figure 17.2.2: The point $\sigma_{n+1} \in C^{\text{sm}}$ lies on a copy of $\mathbf{P}^1$ that transversally meets one other irreducible component Y at 2 smooth points of Y .

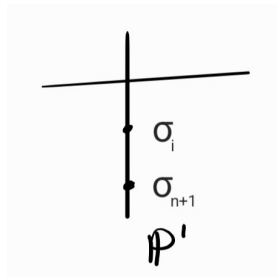


Figure 17.2.3: The point $\sigma_{n+1} \in C^{\text{sm}}$ lies on a copy of $\mathbf{P}^1$ that contains another marked point σ_i and transversally meets one other irreducible component Y at one smooth point of Y .

To fix this, the geometric idea is to “contract” the problematic copy of $\mathbf{P}^1$ (carrying along the section σ_{n+1} to the point onto which the offending $\mathbf{P}^1$ is contracted). One challenge is that a construction achieving this on geometric fibers must be built in a manner that is functorial with respect to isomorphisms and naturally compatible with arbitrary base change. Here are the desired geometric contractions over an algebraically closed field, corresponding to the 3 preceding Figures:

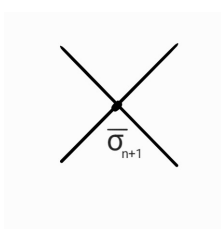


Figure 17.2.4: Contract $\mathbf{P}^1$ to a point $\bar{\sigma}_{n+1}$ and glue Y and Y' where each met the $\mathbf{P}^1$.

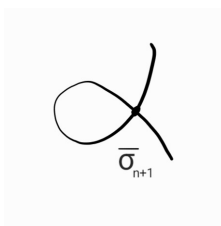


Figure 17.2.5: Contract $\mathbf{P}^1$ to a point $\bar{\sigma}_{n+1}$ and glue Y to itself at the two smooth points where it met the $\mathbf{P}^1$.

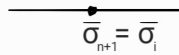


Figure 17.2.6: Contract $\mathbf{P}^1$ to the smooth point where it meets Y , and label that as $\bar{\sigma}_{n+1}$ and $\bar{\sigma}_i$.

To perform the contraction operation intrinsically, Knudsen's idea is as follows. Recall from [EGA, II, 4.6.3(b)] (applied in the proper case, and using that a dominant open immersion between proper schemes is an isomorphism) and the miraculous [EGA, IV₃, 9.9.6] that for any proper finitely presented map $f : X \rightarrow S$ and any fiberwise ample line bundle $\mathcal{L}$ on X , we have naturally

$$X \simeq \mathrm{Proj}_S(\oplus_m f_*(\mathcal{L}^{\otimes m})).$$

We apply this with $X = \mathcal{C}$ a stable $(n+1)$ -pointed genus- g curve (where $2g - 2 + n > 0$) having markings $\sigma_1, \dots, \sigma_{n+1}$ and the fiberwise-ample

$$\mathcal{L} = \tilde{\omega}_{\mathcal{C}/S} := \omega_{\mathcal{C}/S}(\sigma_1 + \dots + \sigma_{n+1}).$$

The idea is to replace this curve with a *contraction* defined by a modified Proj construction:

$$c(\mathcal{C}) := \mathrm{Proj}_S(\oplus_{m \geq 0} f_*(\mathcal{N}^{\otimes m}))$$

where $\mathcal{N} = \omega_{\mathcal{C}/S}(\sigma_1 + \dots + \sigma_n) = \tilde{\omega}_{\mathcal{C}/S}(-\sigma_{n+1})$. In [Knu83] it is shown that this “works” in the sense that

- its formation commutes with any base change,
- there is a canonical S -map (compatible with base change) $\pi : \mathcal{C} \rightarrow c(\mathcal{C})$ for which the resulting sections $\bar{\sigma}_i = \pi \circ \sigma_i$ with $1 \leq i \leq n$ make

$$(c(\mathcal{C}), \bar{\sigma}_1, \dots, \bar{\sigma}_n)$$

a stable n -pointed genus- g curve,

- on geometric fibers this yields the desired contraction procedure.

Thus, we get a well-posed natural transformation between fibered categories $\overline{\mathcal{M}}_{g,n+1} \rightarrow \mathcal{Z}_{g,n}$ via

$$(\mathcal{C}, \sigma_1, \dots, \sigma_{n+1}) \mapsto (c(\mathcal{C}), \bar{\sigma}_1, \dots, \bar{\sigma}_n; \bar{\sigma}_{n+1}).$$

In [Knu83], an intrinsic relative inverse procedure

$$\mathcal{Z}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n+1}$$

called *stabilization* is also defined (at the level of quasi-coherent graded sheaves of algebras). On geometric fibers, this performs a blow-up for Figures 17.2.4 and 17.2.5 but for Figure 17.2.6 it is more subtle (as blow-up in those cases would have no effect).

Now that we have built an isomorphism $\overline{\mathcal{M}}_{g,n+1} \simeq \mathcal{Z}_{g,n}$ of fibered categories (yielding the composite map $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ defined by $(\mathcal{C}, \sigma_1, \dots, \sigma_{n+1}) \mapsto (c(\mathcal{C}), \bar{\sigma}_1, \dots, \bar{\sigma}_n)$), we can perform induction on n (with fixed g) for smoothness and properness. All we have to show is that $\mathcal{Z}_{g,n}$ is smooth and proper over $\mathbf{Z}$ when $\overline{\mathcal{M}}_{g,n}$ is. The case of properness is clear since the structure map

$$f : \mathcal{Z}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$$

for the universal n -pointed stable genus- g curve is certainly proper.

For $\mathbf{Z}$ -smoothness, all is clear on the open substack of $\mathcal{Z}_{g,n}$ where it is smooth over $\overline{\mathcal{M}}_{g,n}$. What about at the singularities in fibers? We want to show that the “total space” $\mathcal{Z}_{g,n}$ is $\mathbf{Z}$ -smooth there too (even though it isn’t smooth over $\overline{\mathcal{M}}_{g,n}$ there). This is akin to having a surjective map $M' \rightarrow M$ between smooth manifolds that has some mildly “singular” fibers. We will take care of this local task on $\mathcal{Z}_{g,n}$ at the start of next time. Then we can finally return to the proof of deJong’s theorem!

Lecture 18

Alteration to a stable curve I

§18.1 Smoothness of $\overline{\mathcal{M}}_{g,n}$

We will now finish the proof of $\mathbf{Z}$ -smoothness of $\overline{\mathcal{M}}_{g,n}$ via induction on n . From [Knu83], we know $\overline{\mathcal{M}}_{g,n+1} \simeq \mathcal{Z}_{g,n}$ and $f : \mathcal{Z}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ is proper and finitely presented. Since the base cases are settled, we can assume that $\overline{\mathcal{M}}_{g,n}$ is $\mathbf{Z}$ -smooth and want to deduce that $\mathcal{Z}_{g,n}$ is $\mathbf{Z}$ -smooth. The open substack $\mathcal{U} \subset \mathcal{Z}_{g,n}$ given by the relative smooth locus over $\overline{\mathcal{M}}_{g,n}$ certainly inherits $\mathbf{Z}$ -smoothness from that of $\overline{\mathcal{M}}_{g,n}$.

The remaining issue is $\mathbf{Z}$ -smoothness at the points $x \in |\mathcal{Z}_{g,n}| - |\mathcal{U}|$, say over $y \in |\overline{\mathcal{M}}_{g,n}|$. We want to describe the “local structure” of $\mathcal{Z}_{g,n}$ near such x . We will use a description of the completed local ring $\widehat{\mathcal{O}}_{\mathcal{Z}_{g,n},x}$ as an algebra over the completed local ring $\widehat{\mathcal{O}}_{\overline{\mathcal{M}}_{g,n},y}$, but since the DM stacks $\mathcal{Z}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are not schemes (hence not ringed spaces) we need to clarify: what do such “completed local rings” mean?

Let’s now digress briefly to discuss a useful notion of “local ring at a point” on a Deligne-Mumford stack $\mathcal{X}$ at a point $x \in |\mathcal{X}|$. Consider étale scheme covers $X, X' \rightrightarrows \mathcal{X}$ and respective points $\tilde{x} \mapsto x$ and $\tilde{x}' \mapsto x$ in each (at the level of associated topological spaces). For the cartesian diagram

$$\begin{array}{ccc} X \times_{\mathcal{X}} X' & \xrightarrow{\text{ét}} & X' \\ \text{ét} \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \end{array}$$

(whose upper-left member is an algebraic space étale surjective onto X and X') and a topological point $\xi \in |X \times_{\mathcal{X}} X'|$ lying over $\tilde{x} \in |X|$ and $\tilde{x}' \in |X'|$ (this exists for the same reason as in the case of schemes) as visualized in

$$\begin{array}{ccc} \xi & \longmapsto & \tilde{x}' \\ \downarrow & & \downarrow \\ \tilde{x} & \longmapsto & x \end{array}$$

For strictly henselian local rings at these points (such a notion makes sense for algebraic spaces via étale scheme covers), the natural local maps $\mathcal{O}_{X,\tilde{x}}^{\text{sh}} \rightarrow \mathcal{O}_{X \times_{\mathcal{X}} X', \xi}^{\text{sh}}$ and $\mathcal{O}_{X', \tilde{x}'}^{\text{sh}} \rightarrow \mathcal{O}_{X \times_{\mathcal{X}} X', \xi}^{\text{sh}}$ are isomorphisms. Thus, we may define $\mathcal{O}_{\mathcal{X},x}^{\text{sh}} := \mathcal{O}_{X,\tilde{x}}^{\text{sh}}$ for any étale scheme cover $X \rightarrow \mathcal{X}$ and $\tilde{x} \mapsto x$. Its completion $\widehat{\mathcal{O}}_{\mathcal{X},x}^{\text{sh}}$ may sometimes be denoted as $\widehat{\mathcal{O}}_{\mathcal{X},x}$.

In [Knu83, Thm. 2.7], Knudsen described $\widehat{\mathcal{O}}_{\mathcal{Z}_{g,n},x}^{\text{sh}}$ as an algebra over $\widehat{\mathcal{O}}_{\overline{\mathcal{M}}_{g,n},y}^{\text{sh}}$. Via Artin approximation (which we’ll discuss in more detail later), such a description at the level of completed local rings implies (via the use of compatible étale scheme neighborhoods) that $(\mathcal{Z}_{g,n}, x) \rightarrow (\overline{\mathcal{M}}_{g,n}, y)$ has an étale neighborhood in common with

$$\begin{array}{ccccc}
(0, 0, 0) & \in & \mathrm{Spec}(R[t, x, y]/(xy - t)) & \simeq & \mathrm{Spec}(R[x, y]) \\
\downarrow & & \downarrow & & \\
0 & \in & \mathrm{Spec}(R[t]) & &
\end{array}$$

where 0 is over some point of $\mathrm{Spec}(R)$, and $R[t]$ is a finitely generated $\mathbf{Z}$ -algebra such that $\mathrm{Spec}(R[t])$ is $\mathbf{Z}$ -smooth (by induction from having an étale cover that is étale over $\overline{\mathcal{M}}_{g,n}$). Informally, $\mathrm{Spec}(R)$ has an étale cover that is étale over the Cartier boundary of $\overline{\mathcal{M}}_{g,n}$. Since $\mathrm{Spec}(R[t])$ is $\mathbf{Z}$ -smooth, it follows that $\mathrm{Spec} R$ is $\mathbf{Z}$ -smooth and thus $\mathrm{Spec} R[x, y]$ is $\mathbf{Z}$ -smooth. This concludes our proof that $\mathbf{Z}$ -smoothness of $\overline{\mathcal{M}}_{g,n}$ implies the same for $\mathcal{X}_{g,n} \simeq \overline{\mathcal{M}}_{g,n+1}$.

§18.2 Extending a rational map to $\overline{\mathcal{M}}_{g,n}$ via alteration

Having finished our foundational discussion for $\overline{\mathcal{M}}_{g,n}$ over $\mathbf{Z}$, let's return to proving deJong's alterations theorem. When we last tuned in to our regularly scheduled programming, we were in the following situation:

$$\begin{array}{ccccccc}
Z = \cup \sigma_i(Y) & \hookrightarrow & X & \longleftarrow & X_U & \longleftarrow & Z_U = \coprod_{i=1}^n U \\
& \searrow & \downarrow f & & \downarrow & & \\
& & Y & \longleftarrow & U & &
\end{array}$$

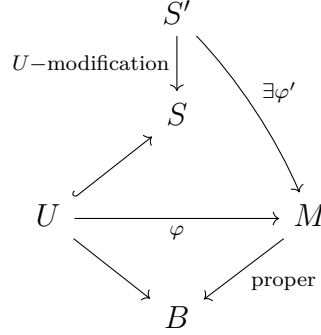
with Y a projective variety over an algebraically closed field k , f a proper k -map from a projective variety X with geometrically connected and generically smooth fibers of pure dimension 1, and $U \subset Y$ a dense open subset over which X_U is U -smooth. Let $g \geq 0$ be the common genus of its fibers. The closed subset $Z \subset X$ is a union of n sections σ_i with properties as in the 3-point lemma, so supported in $\mathrm{sm}(X/Y)$ and meeting at least 3 points in each irreducible component of each geometric fiber. In particular $n \geq 3$, so $2g - 2 + n > 0$.

By shrinking U if necessary we can arrange that the sections $\sigma_i|_U$ are pairwise disjoint since that holds over η_Y ($Z \rightarrow Y$ is generically étale). In particular, the U -smooth X_U equipped with $\underline{\sigma}_U$ is a stable n -pointed genus- g curve over U , so we have a k -map $\varphi : U \xrightarrow{(X_U, \underline{\sigma}_U)} \mathcal{M}_{g,n/k} \subset \overline{\mathcal{M}}_{g,n/k}$ with target a proper DM stack over k .

The pair $(X, \underline{\sigma})$ over Y is typically not stable, since some fibers may not be semistable and some sections may collide (over points in $Y - U$). More broadly, $(X_U, \underline{\sigma}_U)$ likely doesn't extend to a stable pair over Y , or in other words $U \rightarrow \overline{\mathcal{M}}_{g,n/k}$ generally does *not* extend to $Y \rightarrow \overline{\mathcal{M}}_{g,n/k}$. Instead, we aim to get such an extension after a generically étale alteration on Y .

To get some intuition for how this goes let's first review the standard “graph closure” trick for extending a rational map at the cost of a controlled modification of the base. Let B be a noetherian scheme (for us it will be $\mathrm{Spec} k$), S an integral B -scheme of finite type, M a proper B -scheme, and $\varphi : U \rightarrow M$ a B -map. We claim there is a U -modification $q : S' \rightarrow S$ for an integral scheme S' ,

say with $U' = q^{-1}(U) \simeq U$, such that $\varphi \circ q : U' \rightarrow M$ extends to a B -map $\varphi' : S' \rightarrow M$. This is summarized as follows:



To build such a U -modification S' of S , we consider the graph

$$\Gamma_\varphi : U \rightarrow U \times_B M$$

that is a closed immersion since it is a base change of $\Delta_{M/B}$ (with M separated over B). Form its schematic closure $\bar{\Gamma} \subset S \times_B M$, which is integral since U is integral (as S is); this meets the open subscheme $U \times_B M$ in Γ_φ , so $q = \text{pr}_1 : \bar{\Gamma} \rightarrow S$ is a U -modification (it is proper since M is B -proper). In other words, $S' := \bar{\Gamma}$ is a U -modification of S , and $\varphi' = \text{pr}_2 : S' \rightarrow M$ is a B -map whose restriction to $q^{-1}(U) \simeq U$ is φ since $\bar{\Gamma} \cap (U \times_B M) = \Gamma_\varphi$.

In our context of intended application, M is replaced by $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g,n/k}$ that is a finitely presented *proper* Deligne-Mumford stack over k rather than a proper scheme over k . Thus, $\Gamma_\varphi : U \rightarrow U \times \overline{\mathcal{M}}$ is generally *not* a closed immersion but instead is just a finite map (as $\Delta_{\overline{\mathcal{M}}/k}$ is proper and quasi-finite, hence finite). Intuitively, if we try to run the graph closure idea anyway we'd get only a closed substack of $Y \times \overline{\mathcal{M}}$ (and this doesn't relate as nicely to U as in the scheme setting).

Fortunately Chow's lemma for Deligne-Mumford stacks will help us to bypass this difficulty by providing a generically étale alteration $q : Y' \rightarrow Y$ for which $\varphi \circ q : q^{-1}(U) \rightarrow \overline{\mathcal{M}}$ extends to a k -map $Y' \rightarrow \overline{\mathcal{M}}$. In other words, if we pass to such Y' and the strict transform X' of X (this is $(X \times_Y Y')_{\text{red}}$) then we can arrange that there exists a k -morphism $Y \rightarrow \overline{\mathcal{M}}_{g,n/k}$ extending φ .

To be precise, the following result is a global variant for proper DM stacks of the valuative criterion of properness that bypasses the need to appeal (as in [deJ96]) to a theorem of Gabber that normalizations of certain DM stacks finite over $\overline{\mathcal{M}}_{g,n}$ are schemes rather than just stacks.

PROPOSITION 18.2.1. Let S be a noetherian scheme (e.g., the spectrum of a field) and let $\mathcal{M} \rightarrow S$ be a proper Deligne-Mumford stack of finite type. Let X be a reduced and irreducible separated S -scheme of finite type, and $U \subset X$ a dense open subscheme equipped with an S -morphism $h : U \rightarrow \mathcal{M}$.

There exists a generically étale alteration $q : X' \rightarrow X$ from a quasi-projective S -scheme X' and a morphism $X' \rightarrow \mathcal{M}$ whose restriction over the dense open $q^{-1}(U)$ coincides with (in the

sense of being isomorphic to) the morphism $h \circ q : q^{-1}(U) \rightarrow \mathcal{M}$.

In the special case $X = \operatorname{Spec}(R)$ for a complete dvr R with an algebraically closed residue field, this recovers the valuative criterion of properness for proper DM stacks (since a generically étale alteration of $\operatorname{Spec}(R)$ is $\operatorname{Spec}(A')$ for an R -finite domain A' with fraction field K' finite separable over K).

Proof. By Chow's Lemma for schemes [EGA, II, §5.6] applied to the separated finite-type $X \rightarrow S$, we can compose back with a preliminary birational proper map $Y \rightarrow X$ for Y quasi-projective over S and rename Y as X to arrange that X is quasi-projective over S ; this will be used at the end.

The key ingredient is Chow's Lemma for DM stacks in Theorem 16.2.5. To apply this, consider the morphism $(1, h) : U \rightarrow X \times_S \mathcal{M}$ to a DM stack *proper* over X . This factors through $U \times_S \mathcal{M}$, and this factored map $j : U \rightarrow U \times_S \mathcal{M}$ is a base change of the diagonal $\Delta_{\mathcal{M}/S}$ that is proper since $\mathcal{M}$ is S -separated. Since $\mathcal{M}$ is a DM-stack, its proper diagonal over S is finite and more specifically is “an immersion up to étale maps” (as one sees via an étale scheme cover of $\mathcal{M}$). The same then holds for j .

Let $\mathcal{Z}$ be the schematic image of $(1, h)$, a reduced and irreducible closed substack of $X \times_S \mathcal{M}$. In particular, $\mathcal{Z}$ is X -proper (since $\mathcal{M}$ is S -proper) and its underlying space over U coincides with the image of the proper j , so the stacky generic point of $\mathcal{Z}$ is *étale* over the generic point of X .

Now we apply Chow's Lemma for DM stacks to the proper $\mathcal{Z} \rightarrow X$ to get a reduced and irreducible quasi-projective X -scheme X' along with a surjective proper X -morphism $X' \rightarrow \mathcal{Z}$ that is *generically étale*. Restricting over the generic point of $\mathcal{Z}$ and using that $X' \rightarrow \mathcal{Z}$ is generically étale, the function field of X' is finite separable over the function field of X . Hence, $q : X' \rightarrow X$ is a generically étale *alteration*. By design the morphism $X' \rightarrow \mathcal{Z} \rightarrow \mathcal{M}$ has restriction over the preimage of U that coincides with $q^{-1}(U) \rightarrow U \xrightarrow{h} \mathcal{M}$. Since $X' \rightarrow X$ is quasi-projective, and we arranged for $X \rightarrow S$ to be quasi-projective by the initial step with Chow's Lemma for schemes, clearly $X' \rightarrow S$ is quasi-projective. $\square$

The upshot of this is there exists a stable n -pointed genus- g $(\mathcal{C}, \underline{\tau})$ over Y and an isomorphism $\psi : (\mathcal{C}_U, \underline{\tau}_U) \simeq (X_U, \underline{\sigma}_U)$ over U . The intuitive meaning is that this is a “global stable reduction theorem” for $(X_U, \underline{\sigma}_U) \rightarrow U$; i.e., after passing to a generically étale alteration on Y we have

$$\begin{array}{ccc} \Gamma_\psi & \xrightarrow{\text{closed}} & \mathcal{C}_U \times_U X_U \\ & \searrow \simeq & \downarrow \\ & & \mathcal{C}_U \end{array}$$

§18.3 Flatness of graph closure after a modification

GOAL. We want there to be a *morphism* $\tilde{\psi} : \mathcal{C} \rightarrow X$ over Y extending the isomorphism $\psi : (\mathcal{C}_U, \underline{\tau}_U) \simeq (X_U, \underline{\sigma}_U)$ over U . Such a map $\tilde{\psi}$ would satisfy $\tau_i \mapsto \sigma_i$ over Y since that holds over the dense open $U \subset Y$, and $\tilde{\psi}$ is a modification (as it is an isomorphism over U).

If such $\tilde{\psi}$ exists then by Y -flatness of $\mathcal{C}$ its graph would be Y -flat and so be the schematic closure inside $\mathcal{C} \times_Y X$ of its own restriction over U ; i.e., the graph of $\tilde{\psi}$ (if it exists) would have to be the schematic closure $T = \overline{\Gamma}_\psi$ inside $\mathcal{C} \times_Y X$. Equivalently, we want $\text{pr}_1 : T \rightarrow \mathcal{C}$ to be an isomorphism. But this is too much to hope for; a basic obstacle is that T may not even be Y -flat (whereas $\mathcal{C}$ is Y -flat).

As a preliminary step, we will make a U -modification on Y (and pass to the corresponding strict transform on X) so that the graph closure T is Y -flat. (Once this is done, by passing to the normalization of Y we'll be able to harness the 3-point lemma to deduce $T \rightarrow \mathcal{C}$ is an isomorphism.) One approach to such “flattening by blow-up (of the base)” is via a deep and powerful result [RG71, Thm. 5.2.2] of Raynaud and Gruson. We'll argue (as deJong does) in an alternative more concrete way via the graph closure trick with Hilbert schemes (a natural global source of flat families).

To make contact with Hilbert schemes, note that $\mathcal{C} \times_Y X$ is a closed subscheme in some $\mathbf{P}_Y^N$ (since $\mathcal{C}$ and X are projective over k , as $\mathcal{C}$ is projective over Y that is projective over k). Thus, the U -flat $T_U = \Gamma_\psi \simeq \mathcal{C}_U$ that is closed in $\mathbf{P}_U^N$ has some common Hilbert polynomial $\Phi(t) \in \mathbf{Q}[t]$ for all fibers (as U is connected).

Step 0: We seek a modification $Y' \rightarrow Y$ such that the graph closure T' over Y' is Y' -flat with geometrically connected fibers of dimension 1. (Note that each fiber $T'_{y'}$ lies inside the surface $\mathcal{C}_{y'} \times X'_{y'}$.) The U -flat closed subscheme $T_U \subset \mathbf{P}_U^N$ corresponds to a k -morphism $U \rightarrow H := \text{Hilb}_{\mathbf{P}_k^N/k}^\Phi$ whose target is projective. Thus, by the graph closure trick, we get a projective U -modification $Y' \rightarrow Y$ so that after making base change to Y' (and passing to the strict transform X' of X) there is a k -map $Y' \rightarrow H$. Pulling back the universal flat family inside $\mathbf{P}_H^N$, we thereby get a Y' -flat closed subscheme $\mathcal{T}' \subset \mathbf{P}_{Y'}^N$, restricting to $\Gamma_{\psi'}$ over $U' \simeq U$ for $\psi' = \psi_{Y'}$. By Y' -flatness, $\mathcal{T}'$ is the schematic closure of $\mathcal{T}'_U = \Gamma_{\psi'}$, so the graph closure $T' \subset \mathcal{C}_{Y'} \times_{Y'} X'$ is Y' -flat!

To summarize, after an appropriate U -modification of Y we can ensure the closed $T \subset \mathcal{C} \times_Y X$ recovering Γ_ψ over U is Y -flat. Thanks to flatness over the base, any further modification base change on T over Y (such as base change to the normalization of Y) is the schematic closure of the locus over U after that base change. Since $T \rightarrow Y$ is proper and flat with 1-dimensional fibers over U (as $T_U \simeq \mathcal{C}_U$), all fibers are 1-dimensional thanks to:

THEOREM 18.3.1. [EGA, IV₃, 12.2.1(ii)] If $Z \rightarrow S$ is proper, flat, and finitely presented, then $s \mapsto \dim(Z_s)$ is locally constant.

Lastly, we want to make $q : T \rightarrow Y$ between projective varieties over k be its own Stein factorization and so have all fibers geometrically connected (thus of *pure* dimension 1). First base change by the normalization $\tilde{Y} \rightarrow Y$, so we can assume Y is normal. Clearly q is surjective (as $q(T)$

is closed and contains the dense open U), so the linear $\mathcal{O}_Y \rightarrow q_*(\mathcal{O}_T)$ between *coherent* sheaves of algebras (even domains over any non-empty open) is injective. To show $\mathcal{O}_Y \rightarrow q_*(\mathcal{O}_T)$ is an isomorphism it suffices by *normality* of Y to check equality over a dense open in Y . Over U equality holds since $T_U \simeq \mathcal{C}_U = X_U$ is a smooth proper U -curve with geometrically connected fibers.

Lecture 19

Normality and applications

§19.1 Normality of semistable curves

We continue from our setup of last time, with the stable n -pointed genus- g curve fibration $\mathcal{C} \rightarrow Y$, an isomorphism $\psi : \mathcal{C}_U \xrightarrow{\sim} X_U$ over some dense open $U \subset Y$ such that the closure of the graph in the fiber product is flat over Y ; i.e., we have

$$\begin{array}{ccc} T := \overline{\Gamma_\psi} & \longrightarrow & \mathcal{C} \times_Y X \\ & \searrow \text{flat} & \downarrow \\ & & Y \end{array}$$

In particular T is a projective variety since $\Gamma_\psi \simeq \mathcal{C}_U$ is a variety. The Y -map $\text{pr}_1 : T \rightarrow \mathcal{C}$ is a proper birational morphism (recall it's the closure of the graph of an isomorphism on an open set $U \subset Y$!) and we want it to be an isomorphism so then we can extend ψ to a *morphism* (over Y) via

$$\tilde{\psi} := \text{pr}_2 \circ \text{pr}_1^{-1} : \mathcal{C} \xrightarrow{\sim} T \rightarrow X.$$

We will do this in two steps.

- (1) We prove that a semistable curve fibration with smooth generic fiber over connected normal noetherian base (for example, our setup $\mathcal{C} \rightarrow Y$) has total space normal.
- (2) We study the fibers of $T \rightarrow \mathcal{C}$ to infer that the map $\text{pr}_{1,\bar{y}} : T_{\bar{y}} \rightarrow \mathcal{C}_{\bar{y}}$ is quasi-finite for any geometric point $\bar{y} \in Y$.

PROPOSITION 19.1.1. With the notation of the above setup, assuming (1) and (2) above, the projection map $\text{pr}_1 : T \rightarrow \mathcal{C}$ is an isomorphism.

Proof. Given (2), the projection map being quasi-finite on all geometric points tells us pr_1 itself must be quasi-finite. But this projection is proper, hence it is in fact finite. Thus we have a finite birational morphism between projective varieties (recall T was the schematic closure of a graph of a morphism of varieties!), so normality of the target $\mathcal{C}$ implies it is an isomorphism. $\square$

Let's now prove the normality as asserted for step (1) above. We first discuss a toy example to get an appreciation of the result itself.

EXERCISE 19.1.A. Let A be a normal noetherian domain with fraction field K , and pick some nonzero $a \in A$. Then

$$B := A[x, y]/(xy - a),$$

along with the map $\text{Spec}(B) \rightarrow \text{Spec}(A)$ induced by the natural inclusion, is a semistable curve fibration of $\text{Spec}(A)$ with smooth fiber over the generic point. Note that B is A -free (in fact, with A -basis $\{1, x, x^2, \dots, y, y^2, \dots\}$), and so injects into $B \otimes_A K \simeq K[x, 1/x]$ which is a domain, so B is a domain.

Use this information to show that B is integrally closed in its field of fractions by hand (i.e., without appealing to the lemma below). To do this, it is helpful to use the fact [Mat90, Thm. 11.5] that if the integrally closed domain A is not a field then $A = \bigcap_{\text{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$ with such $A_{\mathfrak{p}}$ a dvr (that enables reducing to the case when A is a dvr, so a is a unit multiple of a power of a uniformizer).

The normality of $\mathcal{C}$ is inherited from Y by:

LEMMA 19.1.2. If $X \rightarrow S$ is a semistable curve with connected normal noetherian base S and smooth generic fiber then X is normal.

Proof. Since S is normal and connected, it is irreducible (as there are only finitely many irreducible components and they cannot meet when the local rings are domains); let η be the unique generic point.

It suffices to verify normality of all the local rings $\mathcal{O}_{X,x}$, for which we will use Serre's homological “ $R_1 + S_2$ ” criterion [Mat90, Thm. 23.8]:

(R_1) If $\dim(\mathcal{O}_{X,x}) \leq 1$ then $\mathcal{O}_{X,x}$ is a field or a dvr.

(S_2) (given that R_1 holds) If $\dim(\mathcal{O}_{X,x}) \geq 2$ then this local noetherian ring has depth at least 2: its maximal ideal contains a regular sequence of length ≥ 2 .

We know that these conditions are satisfied by $\mathcal{O}_{S,s}$ for all $s \in S$. We will use this fact in conjunction with the dimension formula for flat maps [Mat90, Thm. 15.1(ii)]:

$$\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{S,f(x)}) + \dim(\mathcal{O}_{X_{f(x)},x}).$$

In particular if $\dim(\mathcal{O}_{S,f(x)}) \geq 2$ then $\dim(\mathcal{O}_{X,x}) \geq 2$, and in such cases there exists a regular sequence of length 2 in the maximal ideal of $\mathcal{O}_{S,f(x)}$, which is also a regular sequence in the maximal ideal of $\mathcal{O}_{X,x}$ by flatness. Thus, the S_2 condition is satisfied at such x . It remains to verify the cases when $\dim(\mathcal{O}_{S,f(x)}) \leq 1$.

If $\mathcal{O}_{S,f(x)}$ 0-dimensional – i.e., a field – then $f(x)$ is the (unique!) generic point η of S , but by hypothesis the generic fiber X_{η} is smooth, so then $\mathcal{O}_{X,x} = \mathcal{O}_{X_{\eta},x}$ is a field or dvr (since the generic fiber is a smooth curve). Thus the R_1 condition is satisfied in such cases.

The remaining case is when $R := \mathcal{O}_{S,f(x)}$ is 1-dimensional; i.e., a dvr with some uniformizer π . Since we are verifying a local condition it is no loss of generality to localize S at $f(x)$ to assume

$S = \operatorname{Spec}(R)$. The point x lies in the special fiber X_0 , so we have

$$\dim(\mathcal{O}_{X,x}) = 1 + \dim(\mathcal{O}_{X_0,x}) = \begin{cases} 1 & \text{if } x \in X_0 \text{ is a generic point,} \\ 2 & \text{if } x \in X_0 \text{ is a closed point.} \end{cases}$$

The uniformizer π of R is a regular element of the maximal ideal of $\mathcal{O}_{X,x}$ by R -flatness and

$$\mathcal{O}_{X_0,x} = \mathcal{O}_{X,x}/(\pi).$$

If x is a generic point in X_0 then this quotient is a field since x is generic in the reduced curve X_0 . Thus, π must generate the maximal of the local noetherian ring $\mathcal{O}_{X,x}$, so by [Ser79, Ch. I, §2, Prop. 2] it follows that $\mathcal{O}_{X,x}$ is a dvr.

Suppose instead $x \in X_0$ is a closed point. To find a length-2 regular sequence in the maximal ideal of $\mathcal{O}_{X,x}$, since π is one such element it suffices to find a regular element in the maximal ideal of the corresponding quotient $\mathcal{O}_{X,x}/(\pi) = \mathcal{O}_{X_0,x}$. Reducedness for noetherian rings is characterized by “ $R_0 + S_1$ ” [EGA, IV₂, 3.2.1(a) $\Leftrightarrow$ (c)] (this rests on the theory of associated primes and primary decomposition for noetherian rings, a key input being that S_1 for a noetherian ring says exactly there are no “embedded” associated primes for (0): the non-minimal primes are *not* associated primes [Mat90, Thms. 6.1(ii), 6.2]). Thus, by reducedness of X_0 the 1-dimensional local noetherian ring $\mathcal{O}_{X_0,x}$ satisfies S_1 : it has a regular element $\bar{t}$ in its maximal ideal. Letting t be a lift of $\bar{t}$ to $\mathcal{O}_{X,x}$, we get a length-2 regular sequence $\{\pi, t\}$ in the maximal ideal of $\mathcal{O}_{X,x}$ as desired. $\square$

§19.2 Application to construction of $\mathcal{C} \rightarrow X$

To conclude that the proper birational $\operatorname{pr}_1 : T \rightarrow \mathcal{C}$ (with normal target) is an isomorphism, it remains to show quasi-finiteness; equivalently, $\operatorname{pr}_{1,\bar{y}} : T_{\bar{y}} \rightarrow \mathcal{C}_{\bar{y}}$ is quasi-finite for all geometric points $\bar{y}$. This will be a consequence of the following result.

LEMMA 19.2.1 (4.20 in [deJ96]). Consider the decompositions into irreducible components:

$$\begin{array}{ccc} & T_{\bar{y}} = T_1 \cup \cdots \cup T_t & \\ \swarrow \operatorname{pr}_{1,\bar{y}} & & \searrow \operatorname{pr}_{2,\bar{y}} \\ \mathcal{C}_{\bar{y}} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s & & X_{\bar{y}} = X_1 \cup \cdots \cup X_r \end{array}$$

Then

(i) for all X_i , there is a unique index $j = j(i)$ such that $T_j \rightarrow X_i$, and this satisfies

(*) $T_{j(i)} \rightarrow \mathcal{C}_{\bar{y}}$ is non-constant.

Also, there exists an open $V \subset X$ that is fiberwise dense such that $\operatorname{pr}_2^{-1}(V) \xrightarrow{\sim} V$.

- (ii) For all $\mathcal{C}_\alpha$, there exists a unique index $\gamma = \gamma(\alpha)$ such that $T_\gamma \rightarrow \mathcal{C}_\alpha$. Also, there exists an open $W \subset \mathcal{C}$ that is fiberwise dense such that $\mathrm{pr}_1^{-1}(W) \xrightarrow{\sim} W$.

REMARK 19.2.2. The statement $(*)$ in the preceding lemma will rest on $(\mathcal{C}, \tau)$ being stable and (X, σ) satisfying the 3-point lemma. The role of (ii) will be to help prove $(*)$.

Granting Lemma 19.2.1 for a moment, we have:

COROLLARY 19.2.3. The projection map $\mathrm{pr}_{1,\bar{y}} : T_{\bar{y}} \rightarrow \mathcal{C}_{\bar{y}}$ is quasi-finite for every $\bar{y}$.

Proof. Assume for the sake of contradiction the projection map is *not* quasi-finite, so there is some positive-dimensional fiber. In other words, some irreducible 1-dimensional component T_j of $T_{\bar{y}}$ maps to a point in $\mathcal{C}_{\bar{y}}$. By condition (i) of Lemma 19.2.1, such a T_j cannot dominate any $X_i \subset X_{\bar{y}}$ under the projection map, so it must also be sent to a point. But $T_j \subset \mathcal{C}_{\bar{y}} \times X_{\bar{y}}$ with the projection map to each factor a point, so T_j itself must be a point, a contradiction! $\square$

Proof of Lemma 19.2.1 (modulo $()$).* We will first prove (i) without $(*)$, and the argument will be entirely symmetric in the roles of X and $\mathcal{C}$ (no use of stability will be made for this part of the argument), so then by the same method (ii) will also hold. We shall write p_i in place of pr_i .

The map $p_2 : T \rightarrow X$ between projective varieties is proper and dominant (it is an isomorphism over U because $T_U = \Gamma_\psi \simeq X_U$ due to $\psi : \mathcal{C}_U \rightarrow X_U$ being an isomorphism), so it is surjective. Thus, $T_{\bar{y}} \rightarrow X_{\bar{y}}$ is surjective, so as a map between connected curves we see one of the finitely many irreducible components T_j must map onto a chosen X_i of interest.

To show such a T_j is unique, it suffices to find the open $V \subset X$ as sought: if $T_{j'} \rightarrow X_i$ for some $j' \neq j$ then finiteness of $T_j \cap T_{j'}$ ensures all but finitely many points in X_i have at least two distinct points over them under $T_{\bar{y}} \rightarrow X_{\bar{y}}$ (namely one point from each of T_j and $T_{j'}$). In particular, $p_2^{-1}(X_i) \rightarrow X_i$ would have at least 2 points in the fibers over all but finitely many points in X_i . But the open $V \subset X$ we seek has $V \cap X_{\bar{y}}$ dense in $X_{\bar{y}}$ and $p_2^{-1}(V) \simeq V$, so $p_2^{-1}(X_i) \rightarrow X_i$ would be an isomorphism over the dense open $V_{\bar{y}} \cap X_i$, a contradiction (as $p_2^{-1}(X_i) \rightarrow X_i$ has at least 2 points in each fiber over all but finitely many points in X_i). This contradiction rules out the occurrence of such a $j' \neq j$ if we can build the desired V .

Thus, to complete the proof of the lemma apart from condition $(*)$ in (i) (including the proof of (ii)), we seek an open $V \subset X$ such that $V \cap X_{\bar{y}}$ is dense in $X_{\bar{y}}$ for all geometric points $\bar{y}$ of Y and $p_2^{-1}(V) \simeq V$. Observe that for each $\bar{y}$, there are only finitely many possible positive-dimensional fibers under $T_{\bar{y}} \rightarrow X_{\bar{y}}$ (since such a fiber has to contain an irreducible component of the pure 1-dimensional $T_{\bar{y}}$, of which there are only finitely many). The subset

$$\Omega = \{x \in X : \dim p_2^{-1}(x) = 0\} \subset X$$

is *open* by upper semi-continuity of fiber dimension (equivalently “ $\dim \geq 1$ ” as a condition on the fiber $p_2^{-1}(x)$ is closed in the base X for $p_2 : T \rightarrow X$, and all fibers are *non-empty*), so the open $\Omega_{\bar{y}} \subset X_{\bar{y}}$ is *dense* for all $\bar{y}$ since its complement is finite.

Recall that we arranged $X_{\bar{y}}$ is generically smooth for all $\bar{y}$, and since Y is normal (so its completed local rings are *domains* by excellence) we also know that

$$\mathrm{sm}(X/Y)_{\bar{y}} = X_{\bar{y}}^{\mathrm{sm}}$$

for all $\bar{y}$ (not just “ $\subset$ ”) by Lemma 9.2.2. Thus, for the open $V := \Omega \cap \mathrm{sm}(X/Y) \subset X$ the geometric fiber

$$V_{\bar{y}} = \Omega_{\bar{y}} \cap X_{\bar{y}}^{\mathrm{sm}}$$

is dense open in $X_{\bar{y}}$ for all $\bar{y}$.

We shall prove the surjection $p_2^{-1}(V) \rightarrow V$ arising from $p_2 : T \rightarrow X$ is an isomorphism. This proper map is quasi-finite since $V \subset \Omega$, so it is *finite*, and it is birational since over U it is an isomorphism (as $T_U \rightarrow X_U$ is an isomorphism, because $T_U = \Gamma_{\psi}$ for $\psi : \mathcal{C}_U \simeq X_U$). As a map between open subsets of the varieties T and X , to conclude the finite birational $p_2^{-1}(V) \rightarrow V$ is an isomorphism it suffices to check V is normal. Although X might not be normal (we have little control on its singularities over $Y - U$), by design $V \subset \mathrm{sm}(X/Y)$ and so V is Y -smooth. Thus, since Y is normal it follows that V is normal! $\square$

Lecture 20

Alteration to a stable curve II

§20.1 Review of situation

We are currently in the setup

$$\begin{array}{ccccc}
 & T = \overline{\Gamma}_\psi & \xrightarrow{\text{closed}} & \mathcal{C} \times_Y X & \\
 \text{pr}_1 \swarrow & & & & \searrow \text{pr}_2 \\
 \mathcal{C} & & & & X \\
 & \xleftarrow{h} & Y & \xrightarrow{f} & \\
 & \tau \swarrow & & \searrow \sigma & \\
 & & U & &
 \end{array}$$

↑ open

with the following properties:

- the map $\psi : \mathcal{C}_U \xrightarrow{\sim} X_U$ is an isomorphism over $U \subset Y$ of stable n -pointed genus- g curves;
- Y (and hence $\mathcal{C}$, by Lemma 19.1.2) is a normal variety;
- all geometric fibers $T_{\bar{y}}$ are connected curves;
- the pair (X, σ) satisfies the 3-point lemma (Lemma 10.2.2).

Our goal is to construct a morphism $\mathcal{C} \rightarrow X$ extending ψ , so then we may replace X with $\mathcal{C}$. To do this, what remains is to show the following statement in Lemma 19.2.1:

- (*) for each geometric fiber $T_{\bar{y}}$, any irreducible component T_j that surjects onto some irreducible component X_i of $X_{\bar{y}}$ has $\text{pr}_{1,\bar{y}} : T_j \rightarrow \mathcal{C}_{\bar{y}}$ non-constant.

This proof of this will be the content of the following subsection.

§20.2 Proof of condition (*)

For notational simplicity, we denote the projection maps from $T_{\bar{y}}$ to $\mathcal{C}_{\bar{y}}$, $X_{\bar{y}}$ as p_1, p_2 respectively.

Assume for the sake of contradiction that $T_j \rightarrow X_i$ and $p_1(T_j) = \{c\} \in \mathcal{C}_{\bar{y}}$ for some irreducible component T_j of $T_{\bar{y}}$. Recall the 3-point lemma tells us there exist 3 distinct points in the smooth locus of X_i arising from sections; i.e.,

$$x_1 := \sigma_{m_1}(\bar{y}), x_2 := \sigma_{m_2}(\bar{y}), x_3 := \sigma_{m_3}(\bar{y})$$

all in X_i^{sm} . Consider the images of the corresponding sections

$$\tau_{m_1}, \tau_{m_2}, \tau_{m_3} : \bar{y} \rightarrow \mathcal{C}_{\bar{y}}$$

which are 3 distinct points (by stability). In particular, without loss of generality we can say that $\tau_{m_1}(\bar{y}), \tau_{m_2}(\bar{y})$ are *not equal* to $c \in \mathcal{C}_{\bar{y}}$. The key insight here towards a contradiction will be in the following lemma.

LEMMA 20.2.1. With notation as above, the disjoint fibers

$$p_2^{-1}(x_1), p_2^{-1}(x_2) \subset T_{\bar{y}}$$

are both connected chains of irreducible components. Moreover, their images

$$p_1(p_2^{-1}(x_1)), p_1(p_2^{-1}(x_2)) \subset \mathcal{C}_{\bar{y}}$$

are also both connected chains of irreducible components without any irreducible component in common.

Proof. Consider the following cartesian diagram:

$$\begin{array}{ccc} T & \xrightarrow{p_2} & X \\ \uparrow & & \uparrow \\ p_2^{-1}(\text{sm}(X/Y)) & \longrightarrow & \text{sm}(X/Y) \end{array}$$

The top horizontal map is known to be proper birational between varieties, hence so is the bottom horizontal map. Moreover, since the Y -smooth locus of X is smooth over a normal base, it is also normal. Therefore, the bottom map is its own Stein factorization. Thus, all fibers of that map are geometrically connected. In particular, since x_1, x_2 both live in $\text{sm}(X/Y)$, we get that $p_2^{-1}(x_1), p_2^{-1}(x_2)$ are connected, so each is either a single point or a connected chain of irreducible components of $T_{\bar{y}}$.

To rule out the case that either is a single point, first note that both fibers meet the irreducible component T_j since it surjects onto X_i , so since $T_j \rightarrow \mathcal{C}_{\bar{y}}$ has image $\{c\}$ by hypothesis we have $c \in p_1(p_2^{-1}(x_1))$ and $c \in p_1(p_2^{-1}(x_2))$. We also have

$$t_h := (\tau_{m_h}(\bar{y}), \sigma_{m_h}(\bar{y})) \in p_2^{-1}(x_i), \quad h = 1, 2$$

(recall $(\tau_m, \sigma_m) : Y \rightarrow \mathcal{C} \times_Y X$ lands inside the closed subvariety T of the target for all $1 \leq m \leq n$, as this may be checked over the dense open $U \subset Y$ where $\psi \circ \tau_m|_U = \sigma_m|_U$ and $T_U = \Gamma_\psi$).

But $p_1(t_h) = \tau_{m_h}(\bar{y}) \neq c$ for $h = 1, 2$, as we arranged without loss of generality right before the lemma statement. We have shown that for $h = 1, 2$ the *distinct* points $p_1(t_h), c$ lie in $p_1(p_2^{-1}(x_h))$, so $p_1(p_2^{-1}(x_h))$ cannot be a single point and hence it is a connected chain of irreducible components.

The same then holds for each fiber $p_2^{-1}(x_h)$ for $h = 1, 2$. By disjointness these fibers have no irreducible components in common, so their images under p_1 also have no irreducible components in common because (by the already-proved part (2) of Lemma 19.2.1!) *at most one* irreducible component of $T_{\bar{y}}$ can map under p_1 onto a given irreducible component of $\mathcal{C}_{\bar{y}}$. $\square$

By Lemma 20.2.1 there are at least two irreducible components of $\mathcal{C}_{\bar{y}}$ through c : one from $p_1(p_2^{-1}(x_1))$ and one from $p_1(p_2^{-1}(x_2))$. This is not possible for a smooth point of the semistable curve $\mathcal{C}_{\bar{y}}$, so c is a singularity. But then $\tau_3(\bar{y}) \neq c$ since the marked points on a stable curve are always in the smooth locus. Thus, we may apply identical logic as in the proof of the Lemma 20.2.1 to get that $p_1(p_2^{-1}(x_3))$ is also connected chain of irreducible components of $\mathcal{C}_{\bar{y}}$ through c having no such components in common with $p_1(p_2^{-1}(x_1))$ and $p_1(p_2^{-1}(x_2))$. Hence, we thereby get at least 3 distinct irreducible components of $\mathcal{C}_{\bar{y}}$ passing through c . That is impossible since $\mathcal{C}_{\bar{y}}$ is semistable.

§20.3 Induction on the base

Letting $\beta : \mathcal{C} \rightarrow X$ be the Y -map extending $\psi : \mathcal{C}_U \simeq X_U$, we are now in the situation

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\beta} & X & \longleftrightarrow & Z = \cup_{i=1}^n \sigma_i(y) \\ & \searrow h & \nearrow f & & \\ & Y & & & \\ & \nwarrow \tau & \nearrow \sigma & & \end{array}$$

where Z is a union of n sections σ_i supported in $\text{sm}(X/Y)$ and is the support of an effective Cartier divisor (i.e., the zero-scheme of an invertible ideal sheaf), $\beta \circ \tau_i = \sigma_i$ for all $1 \leq i \leq n$, and β_U is an isomorphism (so β is a modification). The last two properties of β imply $\beta^{-1}(Z) \subset (\cup_{i=1}^n \tau_i(y)) \cup h^{-1}(D) =: \mathfrak{Z}$ for the closed subset $D = (Y - U)_{\text{red}} \subsetneq Y$ (on which we have very little control; it may not be a divisor). We also saw in Remark 6.1.2 that we may always replace Z with a bigger proper closed subset, so we may replace (X, Z) with $(\mathcal{C}, \beta^{-1}(Z))$ and then pass to $(\mathcal{C}, \mathfrak{Z})$.

Now at long last we harness our dimension induction carrying along the arbitrary proper closed subset: since $\dim Y = \dim X - 1$ there exists a generically étale alteration $\varphi : Y' \rightarrow Y$ such that Y' is k -smooth and $D' := \varphi^{-1}(D)$ is an sncd in Y' . We then have

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\tilde{\varphi}} & \mathcal{C} \\ h' \downarrow & & \\ Y' & & \end{array}$$

where $\mathcal{C}' := \mathcal{C}_{Y'} \rightarrow Y'$ with $\tau' = \tau_{Y'}$ is a stable curve smooth over $U' = \varphi^{-1}(U)$, and $\tilde{\varphi}$ is a generically étale alteration (since φ is) such that $\mathfrak{Z}' := \tilde{\varphi}^{-1}(\mathfrak{Z}) = (\cup_{i=1}^n \tau'_i(Y')) \cup h'^{-1}(D')$ with D' an sncd in Y' . We can now pass to $(\mathcal{C}', \mathfrak{Z}')$ fibered over the *smooth* Y' in a very nice way; it remain to resolve singularities for such special situations (with the preimage of $\mathfrak{Z}'$ an sncd).

Lecture 21

Artin approximation

§21.1 Motivation via semistable curves

We begin with a model example that we will want to later generalize to a “decompletion” machine via étale neighborhoods.

EXAMPLE 21.1.1. Take R a dvr with uniformizer π , residue field κ , and fraction field K . We define “standard” semistable curves over R to be

$$C_n := \operatorname{Spec}(R[u, v]/(uv - \pi^n)),$$

for $n \geq 1$. This has generic fiber $\operatorname{Spec}(K[u, u^{-1}])$ – which is visibly K -smooth – and special fiber $\operatorname{Spec}(\kappa[u, v]/(uv))$ that is the union of the coordinate axes in $\mathbf{A}_\kappa^2$. This special fiber has precisely 2 irreducible components with a (κ -rational!) singularity $\xi = (u, v)$ that is the unique point at which C_n is *not* R -smooth. Thus, $C_n - \{\xi\}$ is regular for all n and the completion of the local ring of C_n at this point is

$$\widehat{\mathcal{O}_{C_n, \xi}} = \widehat{R}[[u, v]]/(uv - \pi^n).$$

For $n = 1$, we note that $uv - \pi \in \mathfrak{m} - \mathfrak{m}^2$, so this quotient ring is regular (and hence C_1 is regular). But for $n \geq 2$, $uv - \pi^n \in \mathfrak{m}^2$ and so this is not regular since $\dim(\mathfrak{m}/\mathfrak{m}^2) = 3$ (with basis $\{\bar{\pi}, \bar{u}, \bar{v}\}$).

We will see in a later lecture that for $n \geq 2$,

$$\operatorname{Bl}_\xi(C_n) = \begin{cases} \text{regular,} & n = 2, \\ \text{covered by copies of } C_{n-2}, & n \geq 3. \end{cases}$$

This gives us an inductive process of (coordinate-free!) singularity resolution for C_n : blow up at the (finitely many) non-regular points singularities, and then repeat. In particular, such blow-up yields a scheme that is again a semistable R -curve.

In general we’ll use the word “singularity” as a synonym for “non-regular point”; it is a property of a scheme at a point and generally *not* a relative property (as for smoothness over a base).

We want to generalize the above example for general (possibly non-proper) semistable curves $C \rightarrow \operatorname{Spec}(R)$ with smooth generic fiber. In particular, such a C may not be covered by C_n ’s for the Zariski topology, and the non-regular locus of the special fiber may not consist of κ -rational points (and such singularities might involve “self-crossing” of a single irreducible component as for

the nodal plane cubic). Artin approximation will rescue the situation by showing that C_n 's provide an adequate model for making calculations for a general semistable curves over a dvr (with smooth generic fiber) if we work “étale-locally”.

We will also generalize to a higher-dimensional regular base; i.e. when we have a semistable curve (with smooth generic fiber) $X \rightarrow Y$ with $\dim(Y) > 1$, and we will be able to push $\text{sing}(X) := X - \text{Reg}(X)$ into codimension ≥ 3 via an intrinsic blow-up process. This will be done while keeping track of the specified proper closed subset $Z \subsetneq X$.

§21.2 Henselian rings and local-étale maps

We will be carrying out many calculations at the level of completed local rings, but need to relate the conclusions back to the original situation before completion. To appropriately “decomplete” we will use an algebraic substitute for completion called *henselization* that has many of the same properties (apart from successive approximation) stemming from Hensel’s Lemma, and is well-behaved even beyond the noetherian case.

A local map of local rings $A \rightarrow A'$ is called *local-étale* if $A' = B_P$ for an étale A -algebra B and a prime ideal P of B . It is a general fact, somewhat in the spirit of a map from a connected space to a connected covering space being determined by its effect at one point, that if $A \rightarrow A' = B_P$ is local-étale (with B étale over A , and k'/k the finite separable extension of residue fields) and X is a connected A -scheme equipped with a point x_0 over the closed point then an A -map $f : X \rightarrow \text{Spec}(A')$ carrying x_0 to the closed point (if one exists!) is uniquely determined by the induced map on residue fields $k' \rightarrow \kappa(x_0)$. In particular, if $A \rightarrow A'$ is residually trivial (i.e., an isomorphism on residue fields) then there is *at most one* $X \rightarrow \text{Spec}(A')$ over A carrying x_0 to the closed point.

The reason f is determined by its effect on residue fields is that if $f, g : X \rightrightarrows \text{Spec}(A')$ are A -maps carrying x_0 to the closed point and inducing the same map $k' \rightarrow \kappa(x_0)$ on residue fields then the corresponding A -map

$$F = (f, g) : X \rightarrow \text{Spec}(A') \times_{\text{Spec}(A)} \text{Spec}(A') = \text{Spec}(A' \otimes_A A') = \Delta \coprod S'$$

with diagonal $\Delta = \text{Spec}(A')$ splitting off as a clopen subscheme (as $A' = B_P$ with A -étale B) carries x_0 into Δ by the equality of *maps* $k' \rightrightarrows \kappa(x_0)$ on residue fields. The clopen subset $F^{-1}(\Delta) \subset X$ is then non-empty and hence coincides with X by connectedness, so F carries X into Δ ; i.e., $f = g$.

This uniqueness result ensures that the collection of all residually trivial local-étale A -algebras has a *unique* structure of directed system via local A -algebra maps: the uniqueness aspect is clear by the preceding discussion, and for the existence aspect we note that if A', A'' are local-étale over A with residue field k then the local ring of $A' \otimes_A A''$ at the evident k -point is local-étale over A and receives local A -algebra maps from A' and A'' .

DEFINITION 21.2.1. A local ring A is **henselian** if any local-étale $\mathrm{Spec}(A') \rightarrow \mathrm{Spec}(A)$ with trivial residue field extension has a section (and so is an isomorphism, as any section to a separated étale map is clopen and local schemes are connected).

There are many different ways to characterize the henselian condition, equivalences among which can be rather deep (e.g., relying on Zariski’s Main Theorem); [EGA, IV₄, 18.5.9, 18.5.13] has a list of many such equivalent formulations, include a version of Hensel’s Lemma: if $F \in A[T]$ is monic then any monic *coprime* factorization $F_0 = G_0 H_0$ over the residue field lifts to a monic factorization in $A[T]$. The assertions in the following definition/result are proved in [EGA, IV₄, 18.6–18.7].

PROPOSITION 21.2.2 (Henselization). For A a local ring, its **henselization** is the local ring given by

$$A^h := \varinjlim_{A \rightarrow A'} A',$$

where the colimit is taken over the directed system of local-étale A -algebras with trivial residue field extension.

This ring is henselian (essentially by design), and has the universal property that any local map from A to a henselian local ring extends uniquely to A^h . Moreover, henselization preserves noetherianity, excellence, reducedness, regularity, and normality, and it commutes with reduction modulo an ideal (i.e., $A/I \rightarrow A^h/I A^h$ is the henselization of A/I), so it also preserves completion in the noetherian case (i.e. $\widehat{A^h} = \widehat{A}$).

The henselization is an “algebraic” substitute for completion. This is made concrete with the following example:

EXAMPLE 21.2.3. Let R be a dvr with fraction field K and $D \subset \mathrm{Gal}(K_s/K)$ a choice of decomposition group (i.e., D is the stabilizer of a choice of maximal ideal $\mathfrak{m}_s$ of the integral closure $R_s \subset K_s$ of R in K_s). By the transitivity of the action of $\mathrm{Gal}(K_s/K)$ on the set of choices of $\mathfrak{m}_s$, it is equivalent to fix a K -embedding of K_s into a separable closure $\widehat{K}_s$ of $\widehat{K}$ and take D to be the image of the resulting continuous injection $\mathrm{Gal}(\widehat{K}_s/\widehat{K}) \hookrightarrow \mathrm{Gal}(K_s/K)$ (injectivity because the splitting field of a separable polynomial over $\widehat{K}$ is unaffected by slightly changing its non-leading coefficients).

For the integral closure R' of R in the fixed field $K_s^D \subset K$ and the maximal ideal $\mathfrak{m}' = R' \cap \mathfrak{m}_s$ of R' under the choice of $\mathfrak{m}_s$, the localization $R'_{\mathfrak{m}'}$ is a henselization of R . Explicitly, in terms of a K -embedding of K_s into $\widehat{K}_s$, we have $K_s^D = K_s \cap \widehat{K}$ and $R'_{\mathfrak{m}'}$ is the localization of $R_s \cap \widehat{R}$ at the maximal ideal below $\widehat{\mathfrak{m}}$ (to prove this, carry out the analogous considerations at the level of finite Galois extensions of K inside K_s , noting that a corresponding local ring on the integral closure has $e = f = 1$ – namely, residually trivial and local-étale over R – exactly when its completion agrees with $\widehat{R}$, and take a direct limit over finite Galois extensions inside K_s/K).

The statement of Artin(-Popescu) approximation will make precise the heuristic that results at the level of a completion can be descended with good approximation to the henselization (and thereby to some residually trivial étale neighborhood).

§21.3 Main result and application

The following result was proved by Artin for A a local ring on a finite type algebra over a field or excellent Dedekind domain (such as $\mathbf{Z}$). A nice reference for its proof in these cases is [BLR90, §3.6] (entirely sufficient for our needs, and sufficient for 99.9999% of all applications).

THEOREM 21.3.1 (Artin-Popescu approximation). Let $(A, \mathfrak{m})$ be an excellent noetherian local ring with maximal ideal $\mathfrak{m}$, and $B := A[x_1, \dots, x_n]/(f_1, \dots, f_r)$ a finite-type A -algebra. The canonical map

$$\mathrm{Hom}_A(B, A^{\mathrm{h}}) \rightarrow \mathrm{Hom}_A(B, \hat{A})$$

has dense image in the $\hat{\mathfrak{m}}$ -adic topology. More precisely, given an A -algebra homomorphism $\phi : B \rightarrow \hat{A}$ and $n \geq 1$ as large as we please, there exists an A -algebra map $\psi : B \rightarrow A^{\mathrm{h}}$ such that

$$\psi \equiv \phi \pmod{\hat{\mathfrak{m}}^n}.$$

Artin conjectured the result as stated above for any excellent local noetherian ring, which was later proved by Popescu (*much more difficult* than the cases treated by Artin). References for a proof with general excellent local A are [Swa98] and [SP, Tag 07BW].

REMARK 21.3.2. The “approximation” aspect in the result is just icing on the cake: the real substance is the case $n = 0$: if a finite system of polynomial equations over A has a solution in $\hat{A}$ then it admits *some* solution in A^{h} (and hence also in some residually trivial local-étale A -algebra A'). Indeed, the approximation to a specified n th-order level with $n \geq 1$ can be encoded by expanding the system of polynomial equations appropriately (involving a choice of generators of $\mathfrak{m}$).

REMARK 21.3.3. The excellence hypothesis cannot be removed, since using $B = A[x]/(x^2)$ for a reduced local noetherian A with $\hat{A}$ non-reduced (as can happen without excellence: an example finite flat over a dvr in characteristic $p > 0$ is implicit in [BLR90, §3.6, Ex. 11], also discussed in the Math Overflow answer [here](#)) would yield a contradiction via a nonzero square-zero element in $\hat{A}$ and n large enough so that its reduction modulo $\hat{\mathfrak{m}}^n$ is nonzero.

EXAMPLE 21.3.4. As an interesting application of Artin-Popescu approximation, we can show that if A is a normal excellent local noetherian domain (e.g., the local ring at the origin for affine n -space over a field) then A^h is the “algebraic closure” of A in $\hat{A}$ (i.e., the subset of elements of $\hat{A}$ that are a root of a nonzero element of $A[T]$).

Indeed, local-étale A -algebras A' inherit normality from A and so are integrally closed domains (by connectedness of $\text{Spec}(A')$) whose fraction field is separable finite over $\text{Frac}(A)$, so everything in A^h is algebraic over A . This henselization injects into its own completion (by noetherianity) that coincides with $\hat{A}$, so A^h is a subring of the algebraic closure of A in $\hat{A}$. For equality, consider $r \in \hat{A}$ that is a root of a non-constant (perhaps non-monic) polynomial $f \in A[T]$. We want to show $r \in A^h$ inside $\hat{A}$.

Since $\hat{A}$ is a domain (as it is local and inherits normality from that of A by excellence), f has only finitely many roots in $\hat{A}$, say $r = a'_1, \dots, a'_m$. For sufficiently large N , the pairwise distinct elements a'_i in $\hat{A}$ are distinct modulo $\hat{\mathfrak{m}}^N$. But Artin approximation provides a root ρ of f in A^h that is congruent to r modulo $\hat{\mathfrak{m}}^N$. The image of ρ in $\hat{A}$ cannot equal any among $a'_2, \dots, a'_m$ since these are all different from $a'_1 = r$ modulo $\hat{\mathfrak{m}}^N$. Thus, ρ has image in $\hat{A}$ that must coincide with r on the nose, so $r \in A^h$ inside $\hat{A}$ as desired.

One can replace A in the target of Hom’s in Artin-Popescu approximation with any local finite-type A -algebra C : the map

$$\text{Hom}_A(B, C^h) \rightarrow \text{Hom}_A(B, \hat{C})$$

has dense image. This follows from the observation that naturally

$$\text{Hom}_A(B, C^h) = \text{Hom}_C(C \otimes_A B, C^h), \quad \text{Hom}_A(B, \hat{C}) = \text{Hom}_C(C \otimes_A B, \hat{C}),$$

and using the statement of Artin-Popescu approximation over C (since $C \otimes_A B$ is a finite-type C -algebra). Since C^h is a direct limit of residually trivial local-étale C -algebras C' , it follows that any solution in $\hat{C}$ to a finite system of polynomial equations over A is approximated arbitrarily closely by a solution in some such C' (note that $\hat{C} = \widehat{C'}$).

For our purposes, the following application of Artin-Popescu approximation is crucial:

PROPOSITION 21.3.5. Let A be an excellent local ring and C_1 and C_2 local A -algebras essentially of finite type (i.e., local rings on finite type A -schemes) with local structure maps $A \rightrightarrows C_1, C_2$.

If $\hat{C}_1 \simeq \hat{C}_2$ as A -algebras then there exists a common residually trivial local-étale algebra C over C_1 and C_2 .

Before proving this consequence, we discuss a very interesting application upon taking A to be a field k :

EXAMPLE 21.3.6. Let X_1, X_2 be finite type schemes over a field k , and $x_i \in X_i(k)$ rational points (arbitrary singularities!). If there is a k -algebra isomorphism of the completed local rings

$$f : \mathcal{O}_{X_2, x_2}^\wedge \simeq \mathcal{O}_{X_1, x_1}^\wedge$$

then we claim there exists a residually trivial local-étale algebra C' over $\mathcal{O}_{X_1, x_1}$ and a local-étale local k -algebra map $\varphi : \mathcal{O}_{X_2, x_2} \rightarrow C'$ such that

$$\mathcal{O}_{X_2, x_2}^\wedge \xrightarrow{\widehat{\varphi}} \widehat{C'} \simeq \mathcal{O}_{X_1, x_1}^\wedge$$

agrees with f modulo $\mathfrak{m}_{x_1}^n$ for whatever $n \geq 1$ we wish.

By spreading out C' to a residually trivial pointed étale neighborhood (X'_2, x'_2) of (X_2, x_2) (i.e., $\mathcal{O}_{X'_2, x'_2} \simeq C'$ over k) and passing to a Zariski-open around x'_2 (renamed as X'_2) if necessary, we get a diagram of étale maps of pointed k -schemes

$$\begin{array}{ccc} (X_1, x_1) & \longleftarrow & (X'_2, x'_2) \\ & & \downarrow \\ & & (X_2, x_2) \end{array}$$

In other words, (X_1, x_1) and (X_2, x_2) have a common residually trivial pointed étale neighborhood (X'_2, x'_2) !

Note that this common étale neighborhood does not “explain” the *specific* isomorphism f between completed local rings (over k), but it nearly does: the resulting k -algebra isomorphism

$$\mathcal{O}_{X_2, x_2}^\wedge \simeq \mathcal{O}_{X'_2, x'_2}^\wedge = \widehat{C'} \simeq \mathcal{O}_{X_1, x_1}^\wedge$$

coincides with $f \bmod \mathfrak{m}_{x_1}^n$.

Let us now prove Proposition 21.3.5:

Proof. We can write $C_2 = B_P$ for a finitely generated A -algebra $B = A[x]/(\underline{f})$ and prime ideal P of B . The given isomorphism f between completed local A -algebras belongs to the set of A -algebra map

$$\mathrm{Hom}_{A, \mathrm{loc}}(\widehat{C_2}, \widehat{C_1}) = \mathrm{Hom}_{A, \mathrm{loc}}(C_2, \widehat{C_1}) = \{F \in \mathrm{Hom}_A(B, \widehat{C_1}) : F^{-1}(\widehat{\mathfrak{m}_1}) = P\}$$

(where the final condition on F encodes the locality requirement after spreading out C_2 to B).

Artin-Popescu approximation says that the latter set contains as a *dense subset* (in terms of congruence modulo high powers of $\widehat{\mathfrak{m}_1}$)

$$\{\varphi \in \mathrm{Hom}_A(B, C_1^{\mathrm{h}}) : \varphi^{-1}(\widehat{\mathfrak{m}_1}) = P\}.$$

But C_1^h is the direct limit of the (uniquely!) directed set of residually trivial local-étale C_1 -algebras C'_1 , and the maximal ideal of each such C'_1 is $\mathfrak{m}_1 C'_1$ (for the maximal ideal $\mathfrak{m}_1$ of C_1) since C'_1 is local-étale over C_1 . Since B is finitely presented over A , we can pass such a direct limit on the target through “Hom” to get the description of this Hom-set as

$$\varinjlim_{C_1 \rightarrow C'_1} \{\varphi \in \operatorname{Hom}_A(B, C'_1) : \varphi^{-1}(\mathfrak{m}_1 C'_1) = P\} = \varinjlim_{C_1 \rightarrow C'_1} \operatorname{Hom}_{A, \text{loc}}(C_2, C'_1).$$

Thus, by renaming some such C'_1 as C_1 (as we may do), we get a local A -algebra map $\varphi : C_2 \rightarrow C_1$ such that $\varphi \equiv f \bmod \widehat{\mathfrak{m}}_1^n$. We need to show that φ is local-étale, or equivalently that $\widehat{\varphi} : \widehat{C}_2 \rightarrow \widehat{C}_1$ is an isomorphism. This completed map may not agree with the given isomorphism f between completions, but by being “closed enough” (congruence modulo the n th power of the maximal ideal with $n \geq 2$) we’ll be able to deduce $\widehat{\varphi}$ is an isomorphism and so be done.

Since $\widehat{\varphi} \equiv f \bmod \widehat{\mathfrak{m}}_1^n$ with $n \geq 2$ and f surjective (even an isomorphism), by successive approximation it follows (check!) that $\widehat{\varphi}$ is surjective. For injectivity, consider the diagram

$$\widehat{C}_1 \xleftarrow{f} \widehat{C}_2 \xrightarrow{\widehat{\varphi}} \widehat{C}_1 \tag{21.3.1}$$

whose first step is an isomorphism and second step is surjective. The composite map from left to right is thereby a surjective *endomorphism* of a noetherian ring, and all such rings maps are isomorphisms by [Mat90, Exer. 3.6] (whose short solution is at the end of the book). Alternatively, we can use length considerations with artinian quotients in this complete local noetherian setting to deduce the isomorphism property for the composite map from left to right in (21.3.1). Either way, with the first step f^{-1} an isomorphism, the isomorphism property for the composite map across (21.3.1) forces the second step $\widehat{\varphi}$ to be an isomorphism as desired. $\square$

Lecture 22

Ordinary Double Point Singularities

§22.1 Definitions and examples

We want to define the notion of “ordinary double point singularity” as a generalization to higher dimensions of the singularities on a semistable curve (this helps to put the latter into a broader context, though in the end we’ll only use it for relative dimension 1). To do so, first we give a characteristic-free definition of non-degeneracy for quadratic spaces over a field (the point being not to treat characteristic 2 in a separate manner; in deJong’s work we want to include characteristic 2, after all):

DEFINITION 22.1.1. For a finite-dimensional vector space V with dimension $n > 0$ over a field k , a non-zero quadratic form $q : V \rightarrow k$ is called *non-degenerate* if the projective quadratic $(q = 0) \subset \mathbf{P}(V) = \text{Proj}(\text{Sym}(V^*)) \simeq \mathbf{P}_k^{n-1}$ is k -smooth.

Note that in the case $n = 1$ that quadric is empty in the point $\mathbf{P}_k^0 = \text{Spec}(k)$, hence smooth, consistent with declaring cx^2 to be non-degenerate for all $c \in k^\times$. A more interesting example of non-degenerate quadratic form is $q(t_1, t_2) = t_1 t_2$ on $V = k^2$, for which the quadric $(q = 0)$ in $\mathbf{P}_k^1$ consists of the two points $[1, 0]$ and $[0, 1]$.

When $\text{char}(k) \neq 2$ or $\dim V$ is even then non-degeneracy of q is equivalent to B_q being perfect, where

$$B_q(v, w) := q(v + w) - q(v) - q(w)$$

is the associated symmetric bilinear form. This is discussed in [Con, HW2, Exercise 4], where one is also led through verifying that over an algebraically closed field there is only one non-degenerate quadratic form on k^n up to isomorphism, given by the following construction (whose non-degeneracy can be directly checked via the Jacobian criterion):

EXAMPLE 22.1.2. The “standard” split quadratic form q_n on k^n is:

$$q_n = \begin{cases} x_1 x_2 + \cdots + x_{n-1} x_n, & n \text{ even}; \\ x_0^2 + q_{n-1}(x_1, \dots, x_{n-1}), & n \text{ odd} \end{cases}$$

(for $n = 1$ we have $q_1 = x_0^2$).

DEFINITION 22.1.3. For a flat finitely present $X \rightarrow S$, a closed point $x \in X_s$ for $s \in S$ is an

ordinary double point singularity if

$$\hat{\mathcal{O}}_{X_{\bar{s}}, \bar{x}} \simeq k(\bar{s})[[t_1, \dots, t_n]]/(q)$$

for a non-degenerate quadratic form q on $k(\bar{s})^n$.

(Here, $\bar{s}$ is a geometric point over s with residue field $\overline{k(s)}$ and $\bar{x}$ is a $k(\bar{s})$ -valued point over x in the geometric fiber $X_{\bar{s}}$. The choice of $\bar{x}$ over x doesn't matter for this definition since $\text{Aut}(k(\bar{s})/k(x))$ acts transitively on the set of such choices even if $k(s)$ or $k(x)$ are imperfect.)

In the case $n = 2$, the preceding recovers the notion of semistable singularity on a curve.

EXAMPLE 22.1.4. We give the “Basic Example” of an ordinary double point singularity. Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k . Let $Q \in A[t_1, \dots, t_n]$ be a residually non-degenerate quadratic form on A^n (i.e. the reduction $\bar{Q}$ of Q modulo $\mathfrak{m}$ is a non-degenerate quadratic form on k^n). Let $S = \text{Spec } A$, $s \in S$ be the closed point corresponding to $\mathfrak{m}$, and $X = \text{Spec}(A[t_1, \dots, t_n]/(Q - a))$ for $a \in \mathfrak{m}$. We claim the finitely presented $X \rightarrow S$ is flat and has an ordinary double point singularity at

$$x = \vec{0}_s \in X_s = \text{Spec}(k[t_1, \dots, t_n]/(\bar{Q})) \subset \mathbf{A}_k^n$$

corresponding to the zero vector in k^n . The geometric fibral local ring at x certainly has the required form, so we just need to check flatness of $X \rightarrow S$.

Writing

$$Q = \sum_{i \leq j} a_{ij} t_i t_j,$$

the residual non-degeneracy of Q implies that $\bar{Q}$ is in particular non-zero, and thus some $a_{ij} \in A^\times$. Thus, the specialization $Q - a \in k(\xi)[t_1, \dots, t_n]$ at any $\xi \in S$ is nonzero and so not a zero-divisor in that polynomial ring. If A is noetherian then it follows from the flatness of polynomial rings and the local flatness criterion for local maps of local noetherian rings [Mat90, Thm. 22.5] (or see [Mat90, Thm. 22.6, Cor.]) that X is S -flat.

Flatness in the general (perhaps non-noetherian) case reduces to the noetherian case by standard “spreading out” arguments. Indeed, the given situation is a base change from an analogue in the noetherian case, namely over the local noetherian ring R_P where $R \subset A$ is the noetherian $\mathbf{Z}$ -subalgebra generated by the a_{ij} 's and a , and P is the contraction of $\mathfrak{m}$ to R .

To push the preceding Basic Example further, we need to investigate its relative smooth locus. Here is a first step:

LEMMA 22.1.5. Let A be a local ring and $Q \in A[t_1, \dots, t_n]$ a residually non-degenerate quadratic form on A^n . For all points in $b \in \text{Spec } A$, the specialization $Q_b \in k(b)[t_1, \dots, t_n]$ at b is non-

degenerate over $k(b)$.

Proof. Let $H = V(Q) \subset \mathbf{P}_A^{n-1}$. Since Q is residually non-degenerate, when Q is written as $\sum_{i \leq j} a_{ij} t_i t_j$ it has a unit coefficient (and so every Q_b is nonzero). Thus H is proper, flat, and finitely presented over $\text{Spec } A$. We will show it is A -smooth, which yields the fibral non-degeneracy.

For $s \in \text{Spec } A$ the closed point, we are given that H_s is $k(s)$ -smooth as Q is residually non-degenerate. Therefore the open A -smooth locus $U \subset H$ contains H_s since A -smoothness at points in the special fiber (or in any fiber) may be checked on the fiber scheme since H is flat and finitely presented over A . This forces $U = H$ by A -properness since an open subscheme V of a proper scheme Y over a local ring must be the entire space if it contains the special fiber (as the closed subset $Y - V \subset Y$ has closed image in the local base that misses the closed point and so must be empty). $\square$

To analyze the non-smooth locus in the Basic Example, we next carry this out over field, which means determining the non-smooth locus in $V(q - c) \subset \mathbf{A}_k^n$ for $c \in k$ and non-degenerate quadratic forms q on a k -vector-space of dimension $n > 0$.

LEMMA 22.1.6. Suppose that k is a field and q is a non-degenerate quadratic form on k^n . Pick $c \in k$, and let $X = \text{Spec}(k[t_1, \dots, t_n]/(q - c))$. The following hold:

- if $c = 0$ then X is k -smooth exactly away from $\vec{0}$;
- if $c \neq 0$, X is k -smooth except when $\text{char}(k) = 2$ with n odd, in which case X is smooth away exactly from a purely inseparable closed point.

Proof. The case for $n = 1$ is easy and is left as an exercise. For $n > 1$ note that a finite type k -scheme Y is smooth over k if and only if $Y_{\bar{k}}$ is $\bar{k}$ -smooth. Thus, we can pass to the case that $k = \bar{k}$. As noted in the discussion preceding Example 22.1.2, there exists a basis $v_1, \dots, v_n$ for k^n such that

$$q(t_1 v_1 + \dots + t_n v_n) = \begin{cases} t_1 t_2 + \dots + t_{n-1} t_n & n \text{ is even,} \\ t_1 t_2 + \dots + t_{n-2} t_{n-1} + t_n^2 & n \text{ is odd.} \end{cases}$$

We can then use the Jacobian criterion to see X is smooth exactly away from the origin when $c = 0$ and is smooth everywhere for $c \neq 0$ except when $\text{char}(k) = 2$ with n odd, in which case smoothness fails at exactly one geometric point. $\square$

Let's see what this says about non-smoothness for the Basic Example. Let A be a local ring, $Q \in A[t_1, \dots, t_n]$ a residually non-degenerate quadratic form on A^n , $S = \text{Spec } A$, and

$$X = \text{Spec}(A[t_1, \dots, t_n]/(Q - a))$$

for $a \in \mathfrak{m}$. This is flat and finitely presented over A , so the smooth locus on X meets each fiber in its own smooth locus over the corresponding residue field. Thus, using Lemma 22.1.6, the closed

non-smooth locus of X is exactly the zero section over $\mathrm{Spec}(A/(a))$ provided that n is even when 2 isn't a unit in A (we will ultimately be interested in the case $n = 2$).

§22.2 Structure Theorem

We are nearly ready to state the étale-local structure theorem for the “total space” near an ordinary double point singularity, provided that the relative dimension is odd when 2 is not a unit on the base. This structure theorem will involve a choice of residually non-degenerate quadratic form over a henselian local ring, so we first note that this choice matters much less than one might think: it only matters through its reduction because any two such quadratic forms with isomorphic (non-degenerate) reductions are isomorphic over the henselian local ring (provided that the number of variables is even when the residue characteristic is 2): this is the main result in Appendix D.

There is one last notion we need to introduce to state the structure theorem: the notions of *strictly henselian* and *strict henselization*. These are the analogues of “henselian” and “henselization” in which we drop the requirement of residual triviality, beginning with:

DEFINITION 22.2.1. A local ring R is called *strictly henselian* if the only local-étale R -algebra is itself.

There are a variety of equivalent characterizations discussed in [EGA, IV₄, §18.8], among which are being henselian with separably closed residue field. For every local ring R with residue field k and a choice of separable closure k_s/k , there is a *strict henselization* R^{sh} with residue field k_s satisfying the universal property that for any local map $f : R \rightarrow A$ to a strictly henselian local ring (inducing a map $k \rightarrow A/\mathfrak{m}$ on residue fields) and a specified k -algebra map $\iota : k_s \rightarrow A/\mathfrak{m}$ on residue fields, there is a unique local factorization of f through $R \rightarrow R^{\mathrm{sh}}$ inducing ι on residue fields.

EXAMPLE 22.2.2. When R is a field, strict henselization is the same as “separable closure”. When R is a complete dvr (or any henselian dvr), strict henselization is the same as “maximal unramified extension” (relative to a choice of separable closure of the residue field).

If R is a local ring with separably closed residue field, its henselization and strict henselization are the same concept.

In general R^{sh} is constructed as a suitably rigidified direct limit of local-étale R -algebras (allowing finite separable residue field extension), discussed and studied with careful attention to the residue field in [EGA, IV₄, §18.8]. In particular, the formation of R^{sh} preserves noetherianity, commutes with passage to the quotient by an ideal, and inherits excellence (though this latter point is accidentally not discussed in [EGA, IV₄, §18.8]; it is explained in [FK88, Ch. 1, pp., 17–18, Rem.]).

REMARK 22.2.3. If A is a local ring and (X_1, x_1) and (X_2, x_2) are two finitely presented A -schemes equipped with points over the closed point of $\text{Spec}(A)$ then we claim any A -algebra isomorphism between strict henselizations

$$f : \mathcal{O}_{X_1, x_1}^{\text{sh}} \simeq \mathcal{O}_{X_2, x_2}^{\text{sh}}$$

necessarily arises from a common étale neighborhood (X', x') of (X_1, x_1) and (X_2, x_2) over A . (Conversely, any such common étale neighborhood certainly yields an A -algebra isomorphism of strict henselizations via the induced isomorphisms $\mathcal{O}_{X_1, x_1}^{\text{sh}} \simeq \mathcal{O}_{X', x'}^{\text{sh}} \simeq \mathcal{O}_{X_2, x_2}^{\text{sh}}$.)

To see this, from the construction of strict henselizations as a direct limit along local-étale (hence injective) transition maps, we may replace (X_2, x_2) by an étale neighborhood so that f arises from a local A -algebra map $\varphi : \mathcal{O}_{X_1, x_1} \rightarrow \mathcal{O}_{X_2, x_2}$ inducing a separable finite extension on residue fields for which the associated map of strict henselizations is the isomorphism f . In this setting, we'll find an open subset $U_2 \subset X_2$ around x_2 so that (U_2, x_2) is an étale neighborhood of (X_1, x_1) over A .

The map φ is flat due to faithful flatness of vertical maps in the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X_1, x_1}^{\text{sh}} & \xrightarrow[\simeq]{f} & \mathcal{O}_{X_2, x_2}^{\text{sh}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{X_1, x_1} & \xrightarrow{\varphi} & \mathcal{O}_{X_2, x_2} \end{array}$$

Likewise, $\Omega_{\mathcal{O}_{X_2, x_2}/\mathcal{O}_{X_1, x_1}}^1 = 0$ because relative Ω^1 is insensitive to étale base change (i.e., $\Omega_{B'/A'}^1 = B' \otimes_A \Omega_{B/A}^1$ for étale $B \rightarrow B'$ over étale $A \rightarrow B'$) and commutes with both localization and direct limits, so

$$\mathcal{O}_{X_2, x_2}^{\text{sh}} \otimes_{\mathcal{O}_{X_2, x_2}} \Omega_{\mathcal{O}_{X_2, x_2}/\mathcal{O}_{X_1, x_1}}^1 \simeq \Omega_{\mathcal{O}_{X_2, x_2}^{\text{sh}}/\mathcal{O}_{X_1, x_1}^{\text{sh}}}^1 = 0$$

(forcing $\Omega_{\mathcal{O}_{X_2, x_2}/\mathcal{O}_{X_1, x_1}}^1 = 0$ by faithful flatness of $\mathcal{O}_{X_2, x_2}^{\text{sh}}$ over $\mathcal{O}_{X_2, x_1}$). By replacing X_2 with a sufficiently small open around x_2 , we can arrange that φ “spreads out” to a map of finitely presented pointed schemes $(X_2, x_2) \rightarrow (X_1, x_1)$ over A . Openness of the flat locus for finitely presented maps then makes $X_2 \rightarrow X_1$ flat near x_2 due to flatness of the induced map φ on local rings, and likewise the finitely presented $\mathcal{O}_{X_2}$ -module Ω_{X_2/X_1}^1 vanishes near x_2 since its x_2 -stalk $\Omega_{\mathcal{O}_{X_2, x_2}/\mathcal{O}_{X_1, x_1}}^1$ vanishes. This provides an open $U_2 \subset X_2$ around x_2 such that the finitely presented $(U_2, x_2) \rightarrow (X_1, x_1)$ over A is flat with vanishing Ω^1 and so is étale.

For a (strictly) henselian local ring $(R, \mathfrak{m})$, we denote the (strict) henselization at the origin in the closed fiber of affine n -space over R (i.e., of $R[T_1, \dots, T_n]$ at $(\mathfrak{m}, t_1, \dots, t_n)$) as

$$R\{T_1, \dots, T_n\}.$$

Finally, we can state the structure theorem:

THEOREM 22.2.4. Consider a scheme S and $X \rightarrow S$ that is flat and finitely presented. Let $x \in X_s$ be an ordinary double point singularity, with $n := 1 + \dim X_s$ required to be even if $\text{char}(k(s)) = 2$. Pick any residually non-degenerate quadratic form $Q \in \mathcal{O}_{S,s}^{\text{sh}}[t_1, \dots, t_n]$ (e.g., $Q = q_n$).

1. There exists $a \in \mathfrak{m}_{\mathcal{O}_{S,s}^{\text{sh}}}$ such that

$$\mathcal{O}_{X,x}^{\text{sh}} \simeq \left(\mathcal{O}_{S,s}^{\text{sh}}[T_1, \dots, T_n] / (Q(T) - a) \right)_{(\vec{0}, s)}^{\text{sh}}$$

as $\mathcal{O}_{S,s}^{\text{sh}}$ -algebras.

Equivalently, by Remark 22.2.3, there exists an affine étale neighborhood $(S' = \text{Spec}(A'), s')$ of (S, s) and $a' \in \mathfrak{p}_{s'}$ carried to a by some $A' \rightarrow \mathcal{O}_{S,s}^{\text{sh}}$ with s' under the closed point such that

$$(X, x), (\text{Spec}(A'[T_1, \dots, T_n] / (q_n(T) - a')), \vec{0}_{s'})$$

have a common étale neighborhood over (S, s) .

2. The ideal $a\mathcal{O}_{S,s}^{\text{sh}}$ is uniquely determined by the preceding conditions, so in particular if this ideal is invertible (hence any principal generator is not a zero-divisor) then a is uniquely determined up to a unit multiple.

Proof. Since the isomorphism of strict henselizations is equivalent to the spread-out version with a common étale neighborhood over (S, s) , by direct limit considerations we can reduce to the case when S is of finite type over $\mathbf{Z}$, so it is noetherian and even excellent (and more specifically of finite type over an excellent Dedekind domain, so we'll only need the version of Artin-Popescu approximation proved by Artin).

We require n to be even when $\text{char}(k(s)) = 2$, so the residual non-degeneracy condition in the definition of ordinary double point singularity at x is equivalent to non-degeneracy of the associated symmetric bilinear form; the latter is the *definition* of “non-degeneracy” used for quadratic forms in [FK88, Ch. III, 2.1]. Thus, the isomorphism of strict henselizations is provided by a rather technical process of descent from a completion using Artin approximation in [FK88, Ch. III, 2.7ff].

The construction of the isomorphism exhibits the ideal $a(\mathcal{O}_{S,s}^{\text{sh}})^\wedge$ in the completion for any such a as an *intrinsic* annihilator ideal, so this ideal is uniquely determined in the completion of $\mathcal{O}_{S,s}^{\text{sh}}$. By faithful flatness of completion in the noetherian case, any two ideals in a local noetherian ring which yield the same ideal in the completion must coincide. This establishes the uniqueness of $a\mathcal{O}_{S,s}^{\text{sh}}$. $\square$

The following consequence (applied in the case $n = 2$) will be of tremendous importance for controlling some étaleness properties when carrying out deJong's resolution process for relative

semistable curves with smooth generic fiber over a regular base:

COROLLARY 22.2.5. For (X, x) an ordinary double point singularity over (S, s) , the finite extension $k(x)/k(s)$ is separable.

EXAMPLE 22.2.6. The separable residue field extension in Corollary 22.2.5 can be arbitrary. Let R be a complete dvr with residue field k and k'/k any finite separable extension with degree d . We claim there exists a proper semistable curve X over $\text{Spec}(R)$ such that its generic fiber is smooth and geometrically connected and its special fiber is geometrically connected and semistable over k with a unique non-smooth point ξ , and $k(\xi) = k'$ over k .

For $d = 1$, a conic with split reduction does the job (e.g., $X^2 + XY + \pi Y^2$ over R for $\pi \in R$ a uniformizer). For $d = 2$, an elliptic curve over $K = \text{Frac}(R)$ with non-split multiplicative reduction splitting over k' (i.e., quadratic twist of a Tate curve over K by the unramified quadratic extension K'/K corresponding to k'/k) does the job.

For general $d > 2$ and infinite k , we give a construction suggested by Ben Church. Let $f(y) \in k[y]$ be the minimal polynomial for a primitive element of k'/k , so $\deg(f) = d$. Let $F(Y, Z) = Y^d + Z(\dots)$ be the homogenization of f , and let $A(Y, Z)$ be the homogenization of a sufficiently “generic” $a(y)$ of degree $d - 2 > 0$ over the infinite field k . In $\mathbf{P}_k^2$ the curve $XZ(X^{d-2} + A(Y, Z)) - F(Y, Z) = 0$ meets $Z = 0$ at $[1, 0, 0]$ which is a smooth point. Away from that point it is the affine curve $x(x^{d-2} + a(y)) = f(y)$ that is checked to be smooth for generic enough a . The reducible curve C_0 over k defined by

$$X(XZ(X^{d-2} + A(Y, Z)) - F(Y, Z)) = 0$$

is thereby seen to have as its geometric singularities $[0, c, 1]$ for geometric roots c of f . This corresponds to a single k' -rational point ξ_0 with residue field k' , and geometrically the two curves $X = 0$ and $XZ(X^{d-2} + A(Y, Z)) = F(Y, Z)$ cross transversally at that point, so this is a semistable over k .

Over R consider the flat curve C in $\mathbf{P}_R^2$ defined by

$$X(XZ(X^{d-2} + \tilde{A}(Y, Z)) - \tilde{F}(Y, Z)) + \pi G(X, Y, Z) = 0$$

for “generic” G of degree d and homogeneous $\tilde{A}$ and $\tilde{F}$ of respective degrees $d - 2$ and d over R lifting A and F over k . This has the preceding semistable special fiber C_0 . By properness over R and the étale-local structure theorem for ordinary double point singularities, C must be semistable over R and thus if its generic fiber isn’t smooth then the singularities arise from R^{sh} -valued points ξ lifting ξ_0 . Such points must lie in $Z \neq 0$ (as we may check on the special

fiber), so they correspond to R^{sh} -points of the Z -dehomogenization

$$x(x(x^{d-2} + \tilde{a}(y)) - \tilde{f}(y)) + \pi g(x, y) = 0$$

that is readily verified to be smooth at all R^{sh} -points lifting $(0, c)$ with $f(c) = 0$. Thus, for “generic” g we have smoothness of the generic fiber.

Alternatively, suppose k is finite or more generally admits a cyclic degree- n extension k'/k . For the corresponding finite unramified extension R' of R with residue field k'/k , the Tate curve $\text{Tate}(\pi^n)$ over R with split n -gon special fiber admits a Galois descent datum corresponding to translation by an n -torsion point that cyclically permutes the n -gon special fiber. The descent is a semistable R -curve whose special fiber is irreducible and has a unique singularity ξ_0 that splits into n distinct rational points over the cyclic k'/k , so $k(\xi_0) = k'$ over k .

REMARK 22.2.7. If $k(x) = k(s)$ then the arguments in [FK88, Ch. III, §2] to prove Theorem 22.2.4 work using just henselization rather than strict henselization (so the “spread out” formulation works with residually trivial étale neighborhoods) subject to one cost: the role of q_n has to be replaced by a residually non-degenerate quadratic form Q over a residually trivial local-étale extension of $\mathcal{O}_{S,s}$ whose isomorphism class we do not control. Note however that any residually non-degenerate Q over $\mathcal{O}_{S,s}^h$ is determined up to isomorphism by its reduction over $k(s)$, by the main result in Appendix D, so the only real loss of control is for the isomorphism class of a quadratic form over $k(s) = k(x)$.

Let’s explain where the latter comes from (in the proof of [FK88, Ch. III, 2.7], which is a bit breezy on the facts about complete intersections which are invoked). From the “geometric” ordinary double point hypothesis over $\overline{k(s)}$, the equality $\mathcal{O}_{X_{\overline{s},\overline{x}}} = \mathcal{O}_{X_{s,x}} \otimes_{k(s)} \overline{k(s)}$ (due to $k(x) = k(s)$) implies that for the complete local $k(s)$ -algebra $\hat{\mathcal{O}}_{X_{s,x}}$ with residue field $k(s)$, its $\mathfrak{m}/\mathfrak{m}^2$ has $k(s)$ -dimension $1 + \dim \mathcal{O}_{X_{s,x}} = 1 + \dim_x X_s = n$ (as such dimension may be computed after scalar extension to $\overline{k(s)}$). Hence, this completion is a quotient of the n -dimensional complete regular local ring $k(s)[[T_1, \dots, T_n]]$ over $k(x) = k(s)$. We claim this is a quotient by a (nonzero) principal ideal.

By [Mat90, Thm. 21.2(i, ii)] it is the same to say that the algebraic local ring $\mathcal{O}_{X_{s,x}}$ is a complete intersection ring. The ordinary double point hypothesis gives that $\hat{\mathcal{O}}_{X_{\overline{s},\overline{x}}}$ is a complete intersection ring, so by [Mat90, Thm. 21.2(i)] the algebraic local ring $\mathcal{O}_{X_{\overline{s},\overline{x}}}$ is a complete intersection ring. But again this ring coincides with $\mathcal{O}_{X_{s,x}} \otimes_{k(s)} \overline{k(s)}$, so by [EGA, IV₄, 19.3.6, 19.3.8, 19.3.9(ii)] it follows that $\mathcal{O}_{X_{s,x}}$ is a complete intersection ring, as desired.

The nonzero relation describing $\hat{\mathcal{O}}_{X_{s,x}}$ as a quotient of $k(s)[[T_1, \dots, T_n]]$ has vanishing linear part due to non-smoothness at x , and we claim its quadratic part q is non-degenerate. That will then be the quadratic form over $k(s)$ whose arbitrary lift over $\mathcal{O}_{S,s}^h$ (unique up to isomorphism by the main result in Appendix D!) is used in the proof. To verify the non-degeneracy it is

sufficient to check over $\overline{k(s)}$, so upon passing to the quotient by the *cube* of the maximal ideal it suffices (by the ordinary double point hypothesis) to check for an isomorphism

$$h : F[T_1, \dots, T_n]/(q, (\underline{T}^3)) \simeq F[T'_1, \dots, T'_n]/(q', \underline{T}'^3)$$

as F -algebras with nonzero quadratic forms q and q' in n variables over a field F that q is conformal to q' (so if one of these is non-degenerate then so is the other).

Lift h to a local F -algebra mapping $\tilde{h} : F[[\underline{T}]] \rightarrow F[[\underline{T}']]$, so on cotangent spaces $\tilde{h}$ induces the same effect on h and hence is an isomorphism on cotangent spaces, so (by successive approximation and dimension considerations) it is an isomorphism. We may compose h with the inverse of $\tilde{h} \bmod \mathfrak{m}^3$ to thereby reduce to the case where $h(T_i) \equiv T'_i \bmod (\underline{T}')^2$, so $(q'(\underline{T}'), (\underline{T}')^3) = (q(\underline{T}'), (\underline{T}')^3)$ as ideals in $F[[\underline{T}']]$. This makes $q'(\underline{T}')$ a nonzero scalar multiple of $q(\underline{T}')$, so the desired conformality holds.

Lecture 23

Refined structure of semistable singularities

§23.1 Descending below the henselization

Let $X \rightarrow S = \operatorname{Spec}(A)$ be a flat and finitely presented map of schemes with A local, and consider $x \in X_s$ for the closed point $s \in S$ with x an ordinary double point singularity. Theorem 22.2.4 tells us we have the following commutative diagram

$$\begin{array}{ccccc}
 & (X', x') & & & \\
 & \swarrow \quad \downarrow \quad \searrow & & & \\
 (X, x) & & (\operatorname{Spec}(A'), s') & & (\operatorname{Spec}(A'[t_1, \dots, t_n]/(q_n - a')), \vec{0}_{s'}) \\
 \downarrow & \swarrow & \searrow & & \downarrow \\
 (S, s) & & & & (\operatorname{Spec}(A'), s')
 \end{array}$$

Here the diagonal maps are all localizations of étale maps, A' is a local-étale extension of A , we have $n := 1 + \dim(X_s)$, q_n is the split n -variable quadratic form over $\mathbf{Z}$ (Example 22.1.2), and the ideal $a'A'$ is uniquely determined. Essentially, we have a common pointed étale neighborhood of both $X \rightarrow S$ and the “Basic Example” in Example 22.1.4.

Our goal is to get a' to lie in $A := \mathcal{O}(S)$ rather than just in a local-étale extension A' in the case of relative dimension 1 (i.e., $n = 2$). In case A is a dvr (with fraction field K) this is easy since A' is a dvr in which a uniformizer of A is also a uniformizer, so either $a' = 0$ or we can replace a with a unit multiple so it becomes a power of a uniformizer in A . Note that if $a' = 0$ then the open image of the étale map $X' \rightarrow \{q_n = 0\}$ hits the closed point of the zero-section and thus the entire local zero-section, which forces X'_k to have a non-smooth point and thus also for X_K by étaleness of $X' \rightarrow X$. In other words, if X_K is smooth and A is a dvr then a' has a unit multiple that is a power of a uniformizer of A .

To achieve the descent of a' to A up to unit multiple more generally in case of relative dimension 1, we impose a mild requirement of equality of residue fields:

PROPOSITION 23.1.1. Let A be a reduced local noetherian ring with residue field k , and suppose $X \rightarrow \operatorname{Spec}(A)$ is a flat semistable curve with smooth generic fibers and special fiber X_0 . Then for any $x \in X_0 - X_0^{\text{sm}}$ with $k(x) = k$, there exists $a \in \mathfrak{m}_A$ not a zero-divisor such that (X, x) has a common étale neighborhood with

$$\operatorname{Spec}\left(A[u, v]/(uv - a), \vec{0} = (u, v, \mathfrak{m}_A)\right).$$

Furthermore, a is unique up to scaling by a unit.

We are not imposing and properness or fibral connectedness hypotheses (and the assertion is local on the source and target). The residue field condition will be harmless for our later work over a higher-dimensional base. It seems unlikely that the descent of a' into A up to unit multiple can be achieved if we don't require $k(x) = k$.

Proof. Theorem 22.2.4 gives us a local étale extension $A \rightarrow A'$, with $a' \in A'$ giving us the étale-local structure. We want a' to be not a zero divisor, and to come from A up to $(A')^\times$ -scaling. We will do this in two steps.

- (not zero-divisor): We will check that a' is non-zero at the generic points of A' , which will then imply a' cannot be a zero-divisor since A' inherits reducedness from A (as it is local-étale over A).
- (descent): We will descend the ideal $a'A'$ to an ideal $I \subset A$. Then $I \otimes_A A' \simeq IA' = a'A'$ being invertible over A' implies (by descent theory, which we can apply by faithful flatness of $A \rightarrow A'$) that I is invertible over the local A , so $I = aA$ by locality of A ; this a is not a zero-divisor in A since $aA = I$ is invertible as an A -module, so it does the job since $aA' = IA' = a'A'$.

(Since we assume $k(x) = k$, by Remark 22.2.7 we could even arrange that $A \rightarrow A'$ is residually trivial. However, this does not simplify anything and so we don't impose it.)

For the first step, by passing to irreducible components of $\text{Spec}(A')$ with their reduced structure (and renaming A' as A and a' as a), our task amounts to showing $a \in \mathfrak{m}_A$ is nonzero. Consider the diagram of the common étale neighborhood given by our structure theorem.

$$\begin{array}{ccc} & (Y, y) & \\ \swarrow & & \searrow \phi \\ (X, x) & & (\text{Spec}(A[u, v]/(uv - a), \vec{0}) \end{array}$$

We will work towards a contradiction by showing if $a = 0$ this would violate the smoothness of the generic fiber X_K .

Assuming $a = 0$, we have the singularity $\xi = (u, v)$ in the generic fiber of $\text{Spec}(A[u, v]/(uv))$. We claim that $\xi \in \text{Im}(\phi)$, so any point in the non-empty $\phi^{-1}(\xi) \subset Y_K$ is not a K -smooth point by étaleness of ϕ ; its image in X_K is then not K -smooth (by étaleness of Y over X), giving us our desired contradiction to the K -smoothness of X_K .

The image $\phi(Y)$ is open in $\text{Spec}(A[u, v]/(uv))$ by étaleness, and it meets the local closed subscheme

$$\text{Spec}(A) \hookrightarrow \text{Spec}(A[u, v]/(uv))$$

(defined by $u = v = 0$) in at least the closed point $\vec{0} \in \text{Spec}(A)$. Since A is local, no proper open subset of its spectrum can contain the closed point, so in fact we must have $\text{Spec}(A) \subset \phi(Y)$. In particular, $\phi(Y)$ contains the generic point ξ of the zero-section $\text{Spec}(A)$ as desired.

Now that we've shown a' isn't a zero-divisor in A' , we need to show its ideal $a'A'$ descends to an ideal in A . Recall the structure theorem tells us the invertible ideal

$$I' := a'A' \subset A'$$

is uniquely determined for our setup $X \rightarrow S$. We have seen that it suffices to find *some* ideal $I \subset A$ such that $I' = A' \otimes_A I (= IA')$. We claim the ideal

$$I := \text{ann}_A(\Omega_{X/A,x}^2) \subset A$$

works. Indeed, by the equality $k(x) = k$ there is a *unique* $x' \in X_{A'}$ over x , so by flatness of $A \rightarrow A'$ we have

$$IA' = A' \otimes_A I = \text{ann}_{A'}(A' \otimes_A \Omega_{X/A,x}^2) = \text{ann}_{A'}(\Omega_{X_{A'}/A',x'}^2).$$

(If we had not assumed $k(x) = k$ then typically $X_{A'}$ would have several points over x and so we have to use the direct sum of the stalks of $\Omega_{X/A}^2$ at all of those points; this would mess up the relation with an annihilator against a stalk at y below.)

Since $Y \rightarrow X_{A'} \rightarrow X$ is étale and $X_{A'} \rightarrow X$ is étale, we have étaleness (hence flatness) of $Y \rightarrow X_{A'}$, so the local $\mathcal{O}_{X_{A'},x'} \rightarrow \mathcal{O}_{Y,y}$ is local-étale. This gives the equality

$$\Omega_{Y/A',y}^2 = \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{X_{A'},x'}} \Omega_{X_{A'}/A',x'}^2,$$

so an $\mathcal{O}_{X_{A'},x'}$ -linear endomorphism T of $\Omega_{X_{A'}/A',x}^2$ vanishes if and only if its effect on $\Omega_{Y/A',y}^2$ vanishes. Thus,

$$A' \otimes_A I = \text{ann}_{A'}(\Omega_{X_{A'}/A',x'}^2) = \text{ann}_{A'}(\Omega_{Y/A',y}^2) = \text{ann}_{A'}(\Omega_{A'[u,v]/(uv-a'),\vec{0}}^2)$$

where the third equality follows from the étaleness of $(Y, y) \rightarrow (\text{Spec}(A'[u, v]/(uv - a')), \vec{0})$ much as the second equality follows from the étaleness of $(Y, y) \rightarrow (X_{A'}, x')$.

Renaming the ring $A'[u, v]/(uv - a')$ as B for notational efficiency, it remains to show the annihilator of $\Omega_{B/A',\vec{0}}^2$ is precisely $I' = a'A'$. To compute this, we first see

$$\Omega_{B/A'}^1 = (B du \oplus B dv) / B(u dv + v du)$$

(since $d_{B/A'}(uv - a') = u dv + v du$). Taking the second exterior power over B and using the behavior of exterior power with respect to quotients gives

$$\Omega_{B/A'}^2 = \wedge^2(B du \oplus B dv) / ((u dv + v du) \wedge \Omega_{B/A'}^1) = (B/(u, v))(du \wedge dv),$$

where we have used the relations $u(du \wedge dv) = -(u dv + v du) \wedge dv = 0$ and $v(du \wedge dv) =$

$(u dv + v du) \wedge dv$). But note that

$$B/(u, v) = (A'[u, v]/(uv - a'))/(u, v) \simeq A'/a'$$

as A' -modules. The quotient $B/(u, v) = A'/(a')$ is local, so taking the stalk of the sheaf at $\vec{0}$ changes nothing, yielding

$$\text{ann}_{A'}(\Omega_{B/A', \vec{0}}^2) = \text{ann}_{A'}(A'/a') = a' A' = I'$$

as desired. $\square$

§23.2 Measure of irregularity over a dvr

As an application of the previous considerations, we have the following corollary when we have a nice 1-dimensional base (i.e., a dvr).

COROLLARY 23.2.1. Take R to be a dvr with uniformizer π , and $X \rightarrow \text{Spec}(R)$ a semistable curve with special fiber X_0 and smooth generic fiber. Then for any $x_0 \in X_0 - X_0^{\text{sm}}$, both (X, x_0) and $(C_n, 0) := \text{Spec}(R[u, v]/(uv - \pi^n), \vec{0})$ share a common pointed étale neighborhood for a uniquely determined $n \in \mathbf{Z}_{\geq 1}$ uniquely determined.

In other words, over a nice 1-dimensional base we always have a common étale neighborhood with our model case described in Example 21.1.1 for a unique n . We denote this as n_{x_0} . Note that there is no requirement for $k(x_0)$ to equal the residue field of R . Since C_n is regular if and only if $n = 1$, and étale morphisms preserve regularity both ways, we have

$$x_0 \in \text{Reg}(X) \iff n_{x_0} = 1.$$

Thus, we call the integer n_{x_0} the **measure of irregularity** at x_0 . We will next analyze the resolution of such points when non-regular:

PROPOSITION 23.2.2. Let X be a semistable curve over a dvr R with fraction field K and smooth generic fiber X_K . Let x_0 be a non-smooth point in the special fiber, and shrink X around x_0 so it is the unique such point. Assume x_0 is not a regular point on X , so its measure of irregularity is some value $n_{x_0} \geq 2$. Let $X' = \text{Bl}_{x_0}(X)$, also R -flat with smooth generic fiber X_K . This is a semistable (so flat) R -curve and:

1. if $n_{x_0} = 2$ or 3 then X' is regular;
2. if $n_{x_0} \geq 4$ then X' has a unique non-regular point x'_0 , it lies over x_0 with $k(x'_0) = k(x_0)$, and $n'_{x'_0} = n_{x_0} - 2 (\geq 2)$.

Note that in (2) we are *not* saying that x'_0 is the only non-smooth point on X' over R , just that

it is the only non-regular point. There generally are other non-smooth points (as is seen in the proof) but they happen to be regular.

The only reason for shrinking X around x_0 to make it the unique non-smooth point is to be able to say X' is regular in part (1) rather than just that X' is regular at the points in its fiber over x_0 (bearing in mind that $X' - X'_{x_0} = X - \{x_0\}$), and likewise that X' has a *unique* non-regular point in part (2). If we remove the uniqueness for x_0 then in (1) and (2) we would have to focus on (ir)regularity at points of X' over x_0 .

Proof. Let π be a uniformizer of R . We want to reduce to the case when $k(x_0) = k$ by carefully increasing R through local-étale extensions without creating more singularities in the special fiber (a possibility if we are too careless in how we enlarge the residue field). We shall carry this out and then further reduce to the case of the standard example C_n from Example 21.1.1 with $n = n_{x_0}$. This special case will be handled next time.

The finite separable extension $k(x_0)/k$ is primitive, so if $f \in k[T]$ is the minimal polynomial for a primitive element and $F \in R[T]$ is a monic lift of f then $R_1 = R[T]/(F)$ is a finite flat R -algebra such that $R_1/(\pi) = k[T]/(f) = k(x_0)$. Hence, R_1 is a discrete valuation ring that is finite étale over R with residue field $k(x_0)$. Let $X_1 = X \otimes_R R_1$ and $Z = \text{Spec}(k(x_0) \otimes_k k(x_0)) \subset (X_1)_{k(x_0)}$, so X_1 is a semistable curve over R_1 (with smooth generic fiber) and the compatibility of blow-up with flat base change gives

$$X' \otimes_R R_1 = \text{Bl}_Z(X_1).$$

Note that the diagonal point x_1 of Z is clopen in Z , so the local structure of $\text{Bl}_Z(X_1)$ at points over $x_1 \in Z$ is identical to that of $X'_1 := \text{Bl}_{x_1}(X_1)$.

Our aim is to determine the local structure of X' at its closed points over x_0 : regularity (or not), semistability, and its residue field at such points that are not regular (if any). We want to relate this to an analogous calculation for a blow-up of X_1 at x_1 , the advantage of the latter being that $k(x_1) = k(x_0)$ is the residue field of R_1 . Due to the definition of $x_1 \in Z$ as a diagonal point, every closed point $x' \in X'$ over $x_0 \in X$ is the image of a unique closed point $\tilde{x}' \in X' \otimes_R R_1$ over $x_1 \in Z$, and we have $\hat{\mathcal{O}}_{X', x'} = \hat{\mathcal{O}}_{X' \otimes_R R_1, \tilde{x}'}$. Thus, to show that X' is semistable over R at all closed points over $x_0 \in X$ (the semistability being obvious elsewhere, since X is semistable over R) it suffices to show the same for X'_1 at all closed points over $x_1 \in X_1$, and the following invariants agree at corresponding points: the residue field (over $k(x_0) = k(x_1)$) and the measure of irregularity. Hence, we conclude that to show when $n_{x_0} \geq 2$ that X' is always semistable over R and is regular at all points over x_0 when $n_{x_0} = 2, 3$ but non-regular at exactly one point over x_0 when $n_{x_0} \geq 4$, with residue field $k(x_0)$ at that point and measure of irregularity $n_{x_0} - 2$ at that point, it suffices to do the same for X'_1 over R_1 at its points over $x_1 \in X_1$. In this way, we may replace (X, x_0, R) with (X_1, x_1, R_1) to reduce to the case $k(x_0) = k$.

Having reduced our task to the case $k(x_0) = k$, by Remark 22.2.7 there is a common *residually trivial* pointed étale neighborhood (Y', y'_0) of (X, x_0) and $(\text{Spec}(R[u, v]/(Q - \pi^n), \bar{0})$ with $n = n_{x_0} \geq 2$

and $Q(u, v)$ some (unknown) residually non-degenerate quadratic form over R . By shrinking Y' around y'_0 to arrange that y'_0 is the *unique* point of Y' over $x_0 \in X$ and over $y_0 \in Y$, so scheme-theoretically

$$Y' \times_X \{x_0\} = \{y'_0\} = Y' \times_Y \{y_0\}.$$

Hence, the compatibility of blow-up with respect to flat (e.g., étale!) base change provides *cartesian* squares

$$\begin{array}{ccccc} \mathrm{Bl}_{x_0}(X) & \longleftarrow & \mathrm{Bl}_{y'_0}(Y') & \longrightarrow & \mathrm{Bl}_{y_0}(Y) \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & Y' & \longrightarrow & Y \end{array}$$

so the étale-local nature of regularity *and* of the measure of irregularity on semistable curves with smooth generic fibers imply that to show $X' := \mathrm{Bl}_{x_0}(X)$ is semistable over R with the desired regularity and residue field properties at its closed points over x_0 it suffices to check the same for $\mathrm{Bl}_{y_0}(Y)$ over R .

Finally, we have to study how the situation for $\mathrm{Bl}_{y_0}(Y)$ over R interacts with local étale extension $R \rightarrow R'$. Note that $Y \otimes_R R'$ has a unique point over y_0 , and $\mathrm{Bl}_{y_0}(Y) \otimes_R R'$ is the blow-up of $Y \otimes_R R'$ at that corresponding point. Hence, if after such extension regularity holds then it held in the first place, and likewise if there is a *unique* non-regular point on the blow-up of $Y \otimes_R R'$ and the residue field at that point agrees with that of R' then the same holds for $\mathrm{Bl}_{y_0}(Y)$ relative to R . (Here we are using that $k(y_0) = k$.) The measure of irregularity is also unchanged by computing on an étale cover, so we see that it suffices to work with $Y \otimes_R R'$ and the unique point on it over y_0 . By taking R' big enough we can arrange that Q is residually split, so Q_{R^h} is isometric to uv (by the main result in Appendix D). But then over some big enough local-étale extension of R we see that Q becomes isometric to uv . Taking that to be R' thereby reduces us to the case $Q = uv$. In other words, we have *finally* reduced to the “Basic Example”

$$(\mathrm{Spec}(R[u, v]/(uv - \pi^n)), \bar{0}).$$

This will be analyzed next time. □

Lecture 24

Blow-up for semistable curves over a dvr I

§24.1 Overview about general blow-up and base change

Before carrying out explicit blow-up calculations on the standard C_n 's over a dvr, let's first review some basic generalities related to blow-up of an affine scheme along a finitely generated ideal. This is a summary of topics discussed in more detail in §A.1–§A.2.

Let A be a ring and $I = (f_1, \dots, f_n)$ a finitely generated ideal. Consider $Z = \operatorname{Spec} A/I \hookrightarrow X = \operatorname{Spec} A$. We want to construct and describe in terms of an affine open cover the blow-up $Y = \operatorname{Bl}_Z(X) = \operatorname{Proj}(\oplus_{n \geq 0} I^n) \subset \mathbf{P}_A^{n-1}$ where the inclusion is by $A[T_1, \dots, T_n] \twoheadrightarrow \oplus I^n$ via $T_j \mapsto f_j$ in degree 1. The blow-up Y has the universal property of being *final* among all X -schemes $\varphi : Y' \rightarrow X$ for which $\varphi^{-1}(Z) \subset Y'$ has invertible ideal sheaf; i.e., $I\mathcal{O}_{Y'}$ is invertible as an $\mathcal{O}_{Y'}$ -module. In terms of a commutative diagram, this says we have a unique factorization

$$\begin{array}{ccc} & Y = \operatorname{Bl}_Z(X) & \\ \nearrow \exists! & \downarrow & \\ Y' & \xrightarrow{\varphi} & X = \operatorname{Spec} A \end{array}$$

when $I\mathcal{O}_{Y'} \subset \mathcal{O}_{Y'}$ is invertible.

For any such Y' , we have an cover by the open subsets

$$Y'_i = \{y' \in Y' : f_i \text{ is a local basis of } I\mathcal{O}_{Y'} \text{ near } y'\}$$

since for any finite list of generators of an invertible sheaf locally one of the generators is a basis (as can be checked in the fiber line $\mathcal{L}(y') = \mathcal{L}_{y'}/\mathfrak{m}_{y'}\mathcal{L}_{y'}$ over the residue field at y . Informally, the condition for a specific f_i to be a local basis is an “open subfunctor” of the condition characterizing the blow-up (this is an informal idea because the blow-up is characterized by a finality property on a subcategory of X -schemes rather than by representing a functor of interest on all X -schemes).

REMARK 24.1.1. The delicate nature of the characterization of blow-up via a finality property rather than a representability property (this will underlie “why” blow-up does not naturally commute with base change) can be appreciated by analogy with another finality characterization one meets earlier in the study of algebraic geometry: normalization. For noetherian integral X , the normalization $\widetilde{X} \rightarrow X$ is final among *normal* integral X -schemes $T \rightarrow X$ with

dominant structure map:

$$\begin{array}{ccc} T & \xrightarrow{\quad \exists! \quad} & \widetilde{X} \\ & \searrow & \swarrow \\ & X & \end{array}$$

where T is normal. If an integral X -scheme T is either not normal or does not have dominant structure map then $T \rightarrow X$ has no reason to factor through $\widetilde{X} \rightarrow X$, and conversely if T is an integral X -scheme equipped with such a factorization then there may be no uniqueness for that if T is not normal or $T \rightarrow X$ is not dominant (so the finality property has nothing to do with $T \rightarrow X$).

We want to directly describe the corresponding open subschemes $Y_i \subset Y = \text{Bl}_Z(X)$ and how Y is built by specific gluing among them. This will be reminiscent of how projective n -space over A is built by gluing $n + 1$ standard affine spaces which represent specific (open) subfunctors of the functor of points of $\mathbf{P}_A^{n-1}$ (namely the condition that the i th member of the specified ordered n -tuple of global sections is a local basis).

To describe Y_i , note that because $(f_1, \dots, f_n)\mathcal{O}_{Y'_i} = I\mathcal{O}_{Y'_i} = f_i\mathcal{O}_{Y'_i}$ which is even *freely generated*, locally we must have $f_j = t_{ij}f_i$ for some local section $t_{ij} \in \mathcal{O}_{Y'_i}$. Hence, we are led to introduce the A -algebras $B_i = A[T_{ij} : j \neq i]/(f_j - T_{ij}f_i)_{j \neq i}$ and $A_i := B_i/(f_i^\infty\text{-torsion})$, where “ f_i^∞ -torsion” denotes the ideal of f_i -power torsion (which we need to kill to make f_i be not a zero-divisor; note that $\mathbf{Z}/(125)$ modulo its 5-torsion has 5 still a zero-divisor). Since $A_i \rightarrow A_i[1/f_i]$ is injective, we may identify A_i with its A -algebra image $A[f_j/f_i]_{j \neq i} \subset A[1/f_i]$ in $A[1/f_i]$. By the above mentioned properties, we have morphisms

$$\begin{array}{ccc} Y'_i & \longrightarrow & \text{Spec } B_i \\ & \searrow & \uparrow \\ & & \text{Spec } B_i/(f_i^\infty\text{-torsion}) \end{array}$$

Thus $Y_i = \text{Spec}(A[f_j/f_i]_{j \neq i})$ is final among A -schemes Y' on which “ f_i is a basis of $I\mathcal{O}_{Y'}$ ” on the category of A -schemes.

EXAMPLE 24.1.2. If $\{f_i\}$ is a regular sequence in A then the ideal of f_i^∞ -torsion in B_i is 0 (so $A_i = B_i$) for each i . This is proved in [FL85, Ch. IV, Thm. 2.2, Cor. 2.5].

We will now describe the gluings of the Y_i ’s. For $i' \neq i$, the overlap $U_{i,i'}$ of Y_i and $Y_{i'}$ inside the blow-up consists of the open subscheme of each characterized by the property that f_i and $f_{i'}$ are *each* a local basis (these are the opens in $\text{Spec}(A_i)$ and $\text{Spec}(A_{i'})$ corresponding to the unit

conditions on $T_{ii'}$ and $T_{i'i}$ respectively). Thus, we have

$$\begin{aligned} \mathbf{A}_A^{n-1} \supset_{\text{closed}} Y_i = \text{Spec}(A[f_j/f_i]_{j \neq i}) &\supset_{\text{open}} \text{Spec}((A[f_j/f_i]_{j \neq i})_{f_{i'}/f_i}) \\ &\simeq \text{Spec}((A[f_j/f_{i'}]_{j \neq i'})_{f_i/f_{i'}}) \subset_{\text{open}} Y_{i'} \subset_{\text{closed}} \mathbf{A}_A^{n-1} \end{aligned}$$

Where the common open $U_{i,i'}$ corresponds is the respective overlap of Y_i and $Y_{i'}$ with the open subschemes of $\mathbf{A}_A^{n-1}$ defined respectively by the unit condition on $T_{ii'}$ and $T_{i'i}$. Thus, the gluing of the n schemes Y_i along the $U_{i,i'}$'s which builds the blow-up Y compatibly arises as a closed subscheme of the gluing $\mathbf{P}_A^{n-1}$ of n copies of affine $(n-1)$ -space over A along the usual overlaps. Hence, we can glue the Y_i 's along opens to make

$$\text{Bl}_Z(X) \subset_{\text{closed}} \mathbf{P}_A^{n-1} = \text{Proj}(A[T_1, \dots, T_n])$$

with $Y_i = Y \cap D_+(T_i)$.

As an application of the finality property characterizing blow-up, we next analyze how it interacts with base change. We have

$$\begin{array}{ccc} Y' = \text{Bl}_{I'}(A) & \xrightarrow{\exists!} & \text{Bl}_I(A) = Y \\ \downarrow & & \downarrow \\ X' = \text{Spec } A' & \longrightarrow & \text{Spec } A = X \\ \uparrow & & \uparrow \\ Z' & \xrightarrow{\quad} & Z \end{array}$$

where $I' := IA'$ and the existence and uniqueness of the top map follows from the fact that $I\mathcal{O}_{Y'} = (IA')\mathcal{O}_{Y'} = I'\mathcal{O}_{Y'}$ is invertible. However, the top commutative square is usually not cartesian! It provides a natural morphism $Y' \rightarrow Y_{A'}$ over A' , but in general there is no reason for $I\mathcal{O}_{Y_{A'}} = I(A' \otimes_A \mathcal{O}_Y)$ to be invertible (hence this may not be an isomorphism) because the map $I \otimes_A A' \rightarrow IA' = I'$ is usually *not* injective. However, it is injective when $A \rightarrow A'$ is flat, so in such cases $Y' \simeq Y_{A'}$ via the natural map. (Concretely in terms of Proj, $Y_{A'}$ is Proj of the graded A' -algebra $\bigoplus (A' \otimes_A I^n)$ and $A' \otimes_A I^n \rightarrow I'^n$ realizes $Y' := \text{Bl}_{I'}(A')$ as a closed subscheme of $Y_{A'}$; there is no reason for this to be an equality with $Y_{A'}$ unless the maps $A' \otimes_A I^n \rightarrow I'^n$ are isomorphisms (such as when $A \rightarrow A'$ is flat))

DEFINITION 24.1.3. For the blow-up $\widetilde{X} = \text{Bl}_Z(X) \xrightarrow{q} X$, we call $\widetilde{Z} := q^{-1}(Z) \subset \widetilde{X}$ the *exceptional divisor* (defined by the invertible ideal sheaf $I\mathcal{O}_{\widetilde{X}}$).

EXAMPLE 24.1.4. Let X be of finite type over a field k and Z a closed point $x \in X$, so $k' := k(x)$ is of finite degree over k . For $\widetilde{X} = \text{Bl}_Z(X) \xrightarrow{q} X$, what is the exceptional divisor $q^{-1}(x)$? The flat base change by $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ commutes with blow-up and leaves the

closed point unchanged (as a closed subscheme), so the exceptional divisor is the same as that of $\text{Bl}_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) = \text{Proj}(\bigoplus_{n \geq 0} \mathfrak{m}_x^n)$. Passing to the special fiber, the exception divisor is Proj of

$$\left(\bigoplus_{n \geq 0} \mathfrak{m}_x^n \right) \otimes_{\mathcal{O}_x} k' = \bigoplus_{n \geq 0} \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}.$$

For example, for $\mathcal{O}_{X,x}$ regular of dimension d we have $\widehat{\mathcal{O}}_{X,x} \simeq k[[t_1, \dots, t_d]]$, so $\bigoplus (\mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}) \simeq \text{Sym}_{k'}(\mathfrak{m}_x / \mathfrak{m}_x^2) = k'[t_1, \dots, t_d]$ as graded k' -algebras. Thus, $q^{-1}(x) \simeq \mathbf{P}_{k'}^{d-1}$. Note this k -scheme is naturally a k' -scheme: it is a projective space over k' . This type of detail is important when making blow-up calculations.

§24.2 Two charts on blow-up of C_n

Let R be a dvr, π a uniformizer, $k = R/\mathfrak{m}$, $A = R[u, v]/(uv - \pi^n)$ with $n \geq 2$. Let $C_n = X = \text{Spec}(A)$ which over R is semistable and smooth away from $\xi = (u, v, \pi) \in X_0$. We want to “calculate” $\text{Bl}_{(u,v,\pi)}(A)$. By the above subsection, it will be covered by $D_+(u), D_+(v), D_+(\pi)$ where $\{u, v, \pi\}$ is a generating set for the ideal along which we are blowing up.

First consider $D_+(u)$. We write the relations $v = v'u$ and $\pi = \pi'u$ where v', π' are new variables corresponding to the “ T_{ij} ”. The A -algebra corresponding to the affine open $D_+(u)$ is

$$(A[v', \pi']/(v - v'u, \pi - \pi'u))/(u^\infty\text{-torsion}),$$

which is the same as $R[u, v, v', \pi']/(uv - \pi^n, v - v'u, \pi - \pi'u)$ modulo u -power torsion.

The relation $v = v'u$ makes v unnecessary, provided we rewrite the relation $uv = \pi^n$ as $v'u^2 = \pi^n$. But we also imposed $\pi = \pi'u$, so

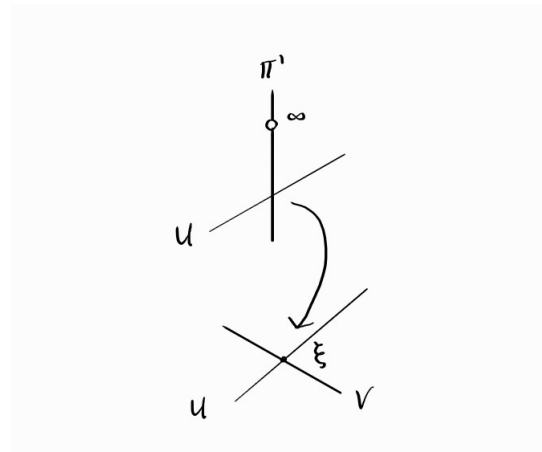
$$v'u^2 = (\pi'u)^n = (\pi')^n u^n.$$

We cancel u^2 from both sides since that can be done modulo u -power torsion, to get $v' = (\pi')^n u^{n-2}$. This makes v' unnecessary too, leaving us with $R[u, \pi']$ modulo the relation $\pi = \pi'u$. But that quotient is a domain, and in particular has vanishing u -power torsion, so

$$D_+(u) = \text{Spec}(R[u, \pi']/(\pi - \pi'u)),$$

which is *regular* (even though its special fiber has a point in the non-smooth locus).

We visualize the special fiber $D_+(u)_k \rightarrow X_0 := \text{Spec}(A/\pi) = \{uv = 0\}$ over $k = R/\pi$ as in Figure 24.2.1 below. (A missing “point at infinity” is drawn on the π' -axis of $D_+(u)_k$ because the gluing with the other two affine open charts of the blow-up will yield a projective line over k containing the π' -axis.) The π' -axis in $D_+(u)_k$ is drawn over the origin singularity ξ in X_0 in that Figure because not only does u restrict to 0 on that line but also v does since $v = v'u = (\pi')^n u^{n-1}$ is divisible by $\pi'u$ which vanishes on $D_+(u)_k$.

Figure 24.2.1: $D_+(u)_k$ over X_0

The computation of $D_+(v)$ goes similarly with $v, \tilde{\pi}'$; we will draw its special fiber over X_0 next time along with how these both interact with the special fiber of $D_+(\pi)$ over X_0 . The details of this calculation are also given in [Liu02, §8.3, Ex. 3.53].

Lecture 25

Blow-up for semistable curves over a dvr II

§25.1 All charts on blow-up of C_n

Recalling our setup from the previous lecture, we had our "Basic Example" of a semistable curve

$$X = C_n = \operatorname{Spec} R[u, v]/(uv - \pi^n).$$

Here R is a dvr with uniformizer π and residue field k . To complete the proof of Proposition 23.2.2, we want to analyze the geometry of the blow-up of X at the unique non-regular point when $n \geq 2$.

$$\xi = (u, v, \pi) \in X.$$

Last time we computed the affine opens $D_+(u), D_+(v) \subset \operatorname{Bl}_\xi(X)$, and it remains to compute $D_+(\pi)$ (and to see how these glue in the special fiber). Before getting to that, we analyze the geometry of the overlap of the special fibers of the two affine opens we've already computed. Although this won't be logically necessary for our specific purposes, the techniques used in the analysis are useful to have in a geometer's toolbox. Recall

$$D_+(u) = \operatorname{Spec} R[u, \pi']/(\pi'u - \pi), \quad D_+(v) = \operatorname{Spec} R[u, \pi'']/(\pi''v - \pi).$$

Here we have $v = (\pi')^n u^{n-1}$ in $D_+(u)$ and $u = (\pi'')^n v^{n-1}$ in $D_+(v)$. Note in particular that $D_+(u)$ and $D_+(v)$ are both *regular* though each has special fiber over k with one non-smooth point.

Defining

$$v' := (\pi')^n u^{n-2}, \quad u' := (\pi'')^n v^{n-2}$$

on $D_+(u)$ and $D_+(v)$ respectively, we have the relations $v = v'u$ in $D_+(u)$ and $u = u'v$ in $D_+(v)$. From the recipe for overlaps of standard affine open charts on a blow-up (namely the unit locus for the corresponding multipliers), the overlap $D_+(u)_k \cap D_+(v)_k$ is precisely the open locus where v' is invertible in $D_+(u)_k = \{u\pi' = 0\} \subset \mathbf{A}_k^2$ (or equivalently, where u' is invertible in $D_+(v)_k$). We split our computations into two cases.

- When $n = 2$, we have $v' = (\pi')^2$, which is invertible precisely away from the origin in the π' -axis in $D_+(u)_k$. Symmetrically this is the locus away from the origin in the π'' -axis in $D_+(v)_k$. Note on this locus we have

$$\pi'\pi'' = \frac{\pi^2}{uv} = 1.$$

Thus in this case the gluing of $D_+(u)_k$ and $D_+(v)_k$ along their overlaps has the π' -axis and

π'' -axis as standard affine lines in a copy of $\mathbf{P}_k^1$.

- When $n \geq 3$, we have $v' = (\pi')^n u^{n-2}$, which is divisible by $\pi' u$ that vanishes on $D_+(u)_k$, so it is invertible nowhere there! Hence, in this case $D_+(u)_k \cap D_+(v)_k = \emptyset$.

To understand the complete picture for the special fiber of $\text{Bl}_\xi(C_n)$ when $n \geq 2$, we need to compute $D_+(\pi)$. Using a similar method as in the previous lecture, we write

$$u = u''\pi, \quad v = v''\pi$$

for new variables u'' and v'' . From this we get

$$\pi^n = uv = u''v''\pi^2 \implies \pi^{n-2} = u''v'',$$

where the cancellation of π^2 is valid since by design π is not a zero divisor on $D_+(\pi)$ (it has vanishing π^∞ -torsion). Putting this all together we have

$$D_+(\pi) = \text{Spec } R[u'', v''] / (u''v'' - \pi^{n-2}) = C_{n-2}$$

with $n - 2 \geq 1$, and on this affine open $u = u''\pi$ and $v = v''\pi$.

Let's go into casework depending on $n \geq 2$ to examine how this open glues to $D_+(u)$, $D_+(v)$ and also the structure of the special fiber.

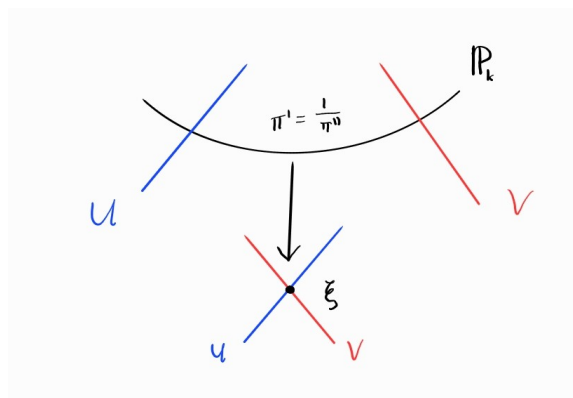
- When $n = 2$, we get that $u''v'' = \pi^{n-2} = 1$, so u'' and v'' are everywhere invertible on the special fiber of $D_+(\pi)$. But the unit condition on such multipliers corresponds to being in $D_+(u)$ and $D_+(v)$ respectively, so $D_+(\pi) \subset D_+(u) \cap D_+(v)$. It follows then that this affine open is redundant for covering the blow-up, so

$$\text{Bl}_\xi(X) = D_+(u) \cup D_+(v),$$

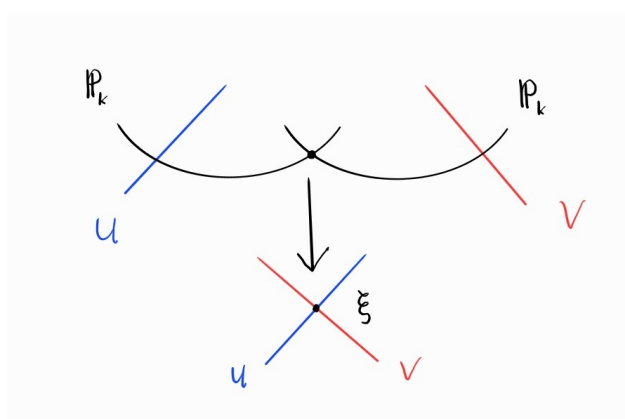
which is regular since each of the two affine open charts is regular by inspection.

Let's check that the special fiber of the blow-up has exceptional divisor isomorphic to $\mathbf{P}_k^1$. We have seen (for $n = 2$) that this exceptional divisor meets $D_+(u)_k$ in the π' -axis and meets $D_+(v)_k$ in the π'' -axis, so it meets the special fiber of $D_+(u) \cup D_+(v) = \text{Bl}_\xi(X)$ in the gluing of such axes away from their origins via the isomorphism $\pi'' = 1/\pi'$. This gluing is exactly a projective line over k , as shown in Figure 25.1.1; remember that *both* singularities in this are regular points on the total space of the blow-up.

- When $n \geq 3$, we have $D_+(\pi)_k = \{u''v'' = 0\}$. This has a single non-smooth point (u'', v'', π) with measure of irregularity $n - 2 > 0$. Since we have seen $D_+(u)$, $D_+(v)$ are regular for all $n \geq 2$, this tells us the total space $\text{Bl}_\xi(X)$ is regular when $n = 3$ and has one non-regular point with measure of irregularity $n - 2$ when $n \geq 4$.

Figure 25.1.1: The special fiber of $D_+(u) \cup D_+(v)$ when $n = 2$

Let's check the the exceptional divisor in this case is isomorphic to a union of two copies of $\mathbf{P}_k^1$ that intersect at a single k -rational point. Indeed, we see that $u'' = u/\pi = 1/\pi'$, and $v'' = v/\pi = 1/\pi''$. Thus the u'' -axis and the π' -axis, minus their respective origins, are identified via the gluing $u'' = 1/\pi''$, which is a projective line over k . Similarly, the identification of the (non-origin) points of the v'' and π'' axes yields a second projective line, and we these two lines intersect in a single k -rational point (u'', v'', π) (corresponding to the singularity in $D_+(\pi)_k$).

Figure 25.1.2: The special fiber of $\text{Bl}_\xi(C_n)$ when $n \geq 3$, with exceptional divisor as a transverse crossing of two $\mathbf{P}_k^1$'s.

We have finally completed the proof of Proposition 23.2.2 by resolving the singularity ξ of the “Basic Example” C_n to which we reduced our task (with $n \geq 2$). In particular, after $\lfloor n/2 \rfloor$ successive blow-ups we reach a regular scheme $X' \rightarrow X = C_n$ for which the fiber X'_ξ is a “chain” of $n - 1$ copies of $\mathbf{P}_k^1$.

§25.2 Scheme structure on the non-smooth locus

Before returning back to the main program, we make a short interlude on the non-smooth and non-regular loci of semistable curves over an arbitrary base scheme.

DEFINITION 25.2.1. An **open semistable curve** over a scheme S is a flat, finitely presented map $f : X \rightarrow S$ whose geometric fibers are all semistable curves.

We distinguish two types of singular loci. First, we have the (closed) **non-smooth locus**

$$\text{sing}(X/S) := X - \text{sm}(X/S),$$

which is a relative notion dependent on the base scheme S over which X lies. This is generally distinct from the *intrinsic* **non-regular locus**

$$\text{sing}(X) := X - \text{Reg}(X).$$

Note that when S is regular we have $\text{sing}(X) \subset \text{sing}(X/S)$. The conventional terminology is slightly unfortunate here – the notion of “singularity” is ambiguous. Unless clear from context one should always specify whether they mean a non-smooth or a non-regular point.

We want to endow the non-smooth locus with a scheme structure with nice properties. It turns out the sheaf of relative 2-forms does exactly this for us.

PROPOSITION 25.2.2. The ideal sheaf

$$\text{ann}_{\mathcal{O}_X}(\Omega_{X/S}^2) \subset \mathcal{O}_X$$

is a finitely generated quasi-coherent sheaf whose formation commutes with any (i.e., not just flat!) base change on S . Its vanishing locus is precisely the non-smooth locus $\text{sing}(X/S)$.

Proof. On the smooth locus $\text{sm}(X/S)$, the sheaf of relative 1-forms is invertible, so its second exterior power vanishes, i.e. $\Omega^2|_{\text{sm}(X/S)} = 0$, so all the desired properties hold trivially.

Away from the smooth locus, we observe that the assertions are étale-local on X and S (here we use that if $h : X' \rightarrow X$ is étale, the natural map $h^*(\Omega_{X/S}^1) \rightarrow \Omega_{X'/S}^1$ is an isomorphism), so it suffices to check them for the “model case”

$$S = \text{Spec}(A), \quad X = \text{Spec } A[u, v]/(uv - a).$$

This case in the affine plane is then handled by a direct calculation that we performed in the proof of Proposition 23.1.1, where we showed the annihilator ideal of this sheaf of 2-forms is exactly (u, v, a) . This is visibly finitely generated, quasi-coherent, and has vanishing locus precisely the

locus $u = v = a = 0$ inside $\text{Spec } A[u, v]/(uv - a)$ (i.e the non-smooth locus). As also shown in that proof, the formation of this ideal commutes with any scalar extension of A . Thus we are done. $\square$

REMARK 25.2.3. The proof shows $\text{ann}_{\mathcal{O}_X}(\Omega_{X/S}^2)$ equals the first **Fitting ideal** $\text{Fitt}_1(\Omega_{X/S}^1) \subset \mathcal{O}_X$. Indeed, the formation of both of these quasi-coherent ideal sheaves commutes with étale base change on X , so it suffices to check their equality in the “model case”, for which it drops out of the calculation of the annihilator ideal in the preceding proof.

§25.3 Return to main program

The setup of our original problem has now been reduced to the following.

SETUP 25.3.1. Let Y be a smooth projective variety of dimension $d-1 \geq 1$ over an algebraically closed field k , and let D be a strict normal crossings divisor in Y . We then have the following setup.

$$\begin{array}{ccc} Z = f^{-1}(D) \cup (\bigcup_{i=1}^n \sigma_i(Y)) & \xrightarrow{\text{closed}} & X \\ & & \downarrow f \\ D & \xrightarrow{\text{sncd}} & Y \end{array}$$

Here $X \rightarrow Y$ is a proper semistable curve with geometrically connected fibers, smooth over $Y - D$. The closed set Z in X is the union of the preimage of D with the image of finitely many sections of Y .

We will put off the desired sncd structure in a resolution of X for the preimage of the closed $Z \subset X$ until the very end; our primary focus for now will be the systematic resolution of singularities for such X as a k -scheme.

For $d = 2$, we have that Y is a curve. Then $\text{sm}(X/Y)$ is the complement of singularities in finitely many fibers. We may then pass to the dvr's $\mathcal{O}_{Y, y_i}$ for $D = \{y_1, \dots, y_m\}$ to see that blowing up X successively at finitely many points eventually reaches a smooth surface. Again note that we still need to analyze the preimage $\tilde{Z}$ of Z in the final resolution, so strictly speaking we aren't yet done with the case $d = 2$ for the induction argument.

For $d \geq 3$, each irreducible component $T \subset \text{sing}(X)$ ($\subset \text{sing}(X/Y) \subset f^{-1}(D)$) has codimension ≥ 2 in X by normality (as X is a semistable curve over the smooth Y and is smooth over the dense open $Y - D \subset Y$). Consider $T \subset \text{sing}(X)$ an irreducible component with codimension 2 (if one exists). In this case we have that $\dim T = d - 2$ and it is over D of pure dimension $d - 2$. It is quasi-finite over D (as even $\text{sing}(X/Y)$ is quasi-finite over Y) and hence finite onto D since f is proper. Thus, we have the finite morphism $T \rightarrow D_i$ for an irreducible component D_i of D (which is regular since D is an sncd in the smooth Y).

Let η_i be the generic point of D_i , so that $\mathcal{O}_{Y,\eta_i}$ is a dvr. We then have that over $\mathcal{O}_{Y,\eta_i}$, $\mathrm{Bl}_T(X)$ is “less non-regular” generically along T . To resolve the puzzle of what happens along the *rest* of T (e.g., is $\mathrm{Bl}_T(X)$ semistable over Y ?), we need to understand the geometry of X near T . This will involve essentially constructing a measure of irregularity for X along the entirety of T .

Lecture 26

Local structure along codimension-2 singularities

§26.1 Setup and main result

We will put the resolution task for $X \rightarrow Y$ in a broader setting (still ignoring Z):

SETUP 26.1.1. Let S be a regular excellent connected noetherian scheme containing a strict normal crossings divisor D . We have the following setup:

$$\begin{array}{ccc} & X & \\ & \downarrow f & \\ D & \xrightarrow{\text{sncd}} & S \end{array}$$

where X is a proper semistable curve with geometrically connected fibers over S . Furthermore, assume f is smooth over $S - D$, $D = \cup_i D_i$ with D_i irreducible (reduced) and hence *regular*.

The ideal sheaves $\mathcal{I}_{D_i} \subset \mathcal{O}_S$ are *invertible* (since $\mathcal{O}_{S,s}$ is regular local and hence a UFD, so all height 1 primes are principal). Also, we have that for $J \subset I$, $D_J := \cap_{i \in J} D_i$ has codimension $\#J$ (by definition of sncd). Excellence will be convenient for passing to $\hat{\mathcal{O}}_{S,s}$ (which is not finite type over k); it is for passing to completions on the base without leaving our axiomatic situation that we have formulated things as generally as above for now.

Note that $\text{sing}(X) := X - \text{reg}(X)$ is closed (by excellence of X) and contained in $\text{sing}(X/S) := X - \text{sm}(X/S)$. For any irreducible component T of $\text{sing}(X)$, we have $\text{codim}_X(T) \geq 2$ since X is *normal*. Last time we showed that $T \rightarrow D_{i_T}$ is finite surjective (for some i_T) if $\text{codim}_X(T) = 2$. We want to “remove” all such T of codimension 2 in X . In the situation,

$$\begin{array}{ccc} \text{Bl}_T(X) & \longrightarrow & X \\ \downarrow & \nearrow f & \\ S & & \end{array}$$

we have that for a generic point η_{i_T} of D_{i_T} , by passing to $\text{Spec}(\mathcal{O}_{S,\eta_{i_T}})$ (which is a dvr) the non-regularity at η_T gets “better” over it in $\text{Bl}_T(X)$.

In our general setup, the normal noetherian X is also connected (since proper flat with connected fibers over S), hence integral. We need to know: is $\text{Bl}_T(X)$ a semistable curve over S ? (Note that it is smooth over $S - D$ since over $S - D$ it is just $X|_{S-D}$ since $T \subset f^{-1}(D)$.)

THEOREM 26.1.2. There exists a modification $\phi : X_1 \rightarrow X$ (i.e. proper birational) with “center” contained in $\text{sing}(X) \subset \text{sing}(X/S)$ – i.e., ϕ is an isomorphism over $\text{Reg}(X)$ – such that

- (1) $X_1 \rightarrow S$ is a semistable curve
- (2) $\text{codim}_{X_1}(\text{sing}(X_1)) \geq 3$.

To prove the theorem, we just have to make sure that $\text{Bl}_T(X)$ is a semistable curve for irreducible components $T \subset \text{sing}(X)$ with codimension $\text{codim}_X(T) \geq 2$. (This does not change $X - T$!). Indeed, for the dvr $R = \mathcal{O}_{Y, \eta_{D_{i_T}}}$ the blow-up $\text{Bl}_{\eta_T}(X_R)$ (localization of $\text{Bl}_T(X)$ at R) has a *unique* non-regular point η' over η_T when the measure of irregularity $n(\eta_T)$ is at least 4, with $n(\eta') = n(\eta_T) - 2$, and is regular when $n(\eta_T) \in \{2, 3\}$. Thus, since $\text{Bl}_T(X) \rightarrow X$ is an isomorphism over $X - T$, this blow-up process does not create new codimension-2 components and improves the situation over one such. In this way, a succession of finitely many such blow-ups would do the job provided semistability over S is preserved.

§26.2 Étale-local structure and completions

We need to understand the étale-local structure of (X, T) for $T \subset \text{sing}(X)$ of codimension 2 in X . We will later see that the finite surjective morphism $T \twoheadrightarrow D_{i_T}$ is étale, so such T are even regular.

Pick a point $x \in X_s^{\text{sing}}$ (e.g., $x \in \text{sing}(X)$, $s = f(x)$). We want to describe $\hat{\mathcal{O}}_{X,x}$ as an $\hat{\mathcal{O}}_{S,s}$ -algebra. Note that $k(x)/k(s)$ is finite separable. Let $D_1, \dots, D_m$ be the irreducible components of D which pass through s . Then the height-1 prime ideal $(\mathcal{I}_{D_i})_s$ is of the form (t_i) for some irreducible $t_i \in \mathfrak{m}_s$ in the regular local ring (hence UFD) $\mathcal{O}_{S,s}$. Our description of $\hat{\mathcal{O}}_{X,x}$ as an $\hat{\mathcal{O}}_{S,s}$ -algebra will be in terms of the t_i 's. Note that in the regular local (hence UFD) completion $\hat{\mathcal{O}}_{S,s}$ the t_i 's remain irreducible since the local quotient $\hat{\mathcal{O}}_{S,s}/(t_i)$ is also regular (hence a domain!) since it is the completion of $\mathcal{O}_{S,s}/(t_i) = \mathcal{O}_{D_i,s}$ that is itself regular local.

Beware that $k(x) \neq k(s)$ may happen, but for the moment suppose we are in the case of equality, so the refined structure theorem of ordinary double point singularities (Remark 22.2.7) gives $\hat{\mathcal{O}}_{X,x} \simeq \hat{\mathcal{O}}_{S,s}[[u, v]]/(Q - a)$ for some residually non-degenerate $Q(u, v)$ over $\hat{\mathcal{O}}_{S,s}$ (which can be taken to have coefficients in $\mathcal{O}_{S,s}$ since any two lifts over the completion of the same non-degenerate binary quadratic form over $k(s)$ are isometric) and $a \in \hat{\mathfrak{m}}_s$. We have two issues:

- we want to relate a to the t_i 's;
- we have to allow $k(x) \neq k(s)$.

Let's first reduce our description task for $\hat{\mathcal{O}}_{X,x}$ over $\hat{\mathcal{O}}_{S,s}$ to the case $k(x) = k(s)$. By the primitive element theorem, we can write $k(x) = k(s)[z]/(F_0)$ for some separable monic irreducible $F_0 \in k(s)[z]$. Let $R = \hat{\mathcal{O}}_{S,s}[z]/(F)$ for a *monic* lift $F \in \hat{\mathcal{O}}_{S,s}[z]$ of F_0 , which is a finite free $\hat{\mathcal{O}}_s$ -module with $R/\hat{\mathfrak{m}}_s R = k(s)[z]/(F_0) = k(x)$ a field. In particular $\hat{\mathcal{O}}_s \rightarrow R$ is finite *étale* (check the discriminant, or equivalently its trace pairing is residually perfect and hence is perfect) with R *local* having residue field $k(x)/k(s)$. Then by Hensel's lemma (for F) applied to $\hat{\mathcal{O}}_{X,s}$ there exists

a unique morphism filling in the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\exists!} & \hat{\mathcal{O}}_{X,s} \\ & \nwarrow & \uparrow \\ & & \hat{\mathcal{O}}_{S,s} \end{array}$$

Now consider the following setup:

$$\begin{array}{ccccccc} \mathrm{Spec}(k(x) \otimes_{k(s)} k(x)) & \hookrightarrow & X'' & \xrightarrow{\quad} & X' & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \bullet & \in & \mathrm{Spec}(R) & \longrightarrow & \mathrm{Spec}(\hat{\mathcal{O}}_{S,s}) & \longrightarrow & S \end{array}$$

where $\bullet$ denotes the closed point in $\mathrm{Spec}(R)$, and we have $x' \in X'$ corresponding to x (via the identification of closed fibers of X' and X). The upper left is a copy of $x \times_s x$ in which the diagonal $\Delta(x)$ is a clopen point x'' since $k(x)/k(s)$ is final separable. Using that $k(x'') = k(x)$, we have:

EXERCISE 26.2.A. The natural local map $\hat{\mathcal{O}}_{X,x} \rightarrow \hat{\mathcal{O}}_{X'',x''}$ is an isomorphism of R -algebras.

Thus, by passing to (X'', x'') we are reduced to the case $k(x) = k(s)$ with $S = \mathrm{Spec}(A)$ for a complete regular local A . In this process, to retain contact with the original t_i 's for the D_i 's through s in the original S (i.e., as generators of distinct height-1 primes in the local ring $\mathcal{O}_{S,s}$) we have to make sure that each t_i remains irreducible in the (necessarily regular local, hence UFD) finite étale extension R of the regular local $\hat{\mathcal{O}}_{S,s}$. This holds because the local quotient $R/t_i R$ is a domain: it even regular, being finite étale over $\hat{\mathcal{O}}_{S,s}/(t_i) = \hat{\mathcal{O}}_{D_i,s}$ that is regular (as D_i is regular).

Next time we will prove:

PROPOSITION 26.2.1. As A -algebras, $\hat{\mathcal{O}}_{X,x} \simeq A[[u, v]]/(Q - t_1^{n_1} \cdots t_m^{n_m})$ with a residually non-degenerate $Q(u, v)$ and n_j not all 0.

The key idea is to show $|\mathrm{Spec}(A/(a))|$ is contained in $|\mathrm{Spec}(A/\prod t_i)| = D$, or equivalently $\mathrm{rad}(a) \supset \mathrm{rad}(\prod t_i)$ in A (a regular UFD). Indeed then $\prod t_i \in \mathrm{rad}(a)$, which implies $a | (\prod t_i)^N$ for some $N \geq 1$ with the t_i 's non-associate irreducibles in the UFD A , forcing a to have as its irreducible factors exactly some of the t_i 's (so a is an $A^\times$ -multiple of some product of powers among the t_i 's). Non-regularity at x corresponds to the conditions $\sum n_j \geq 2$ (i.e., $Q - \prod t_j^{n_j}$ is in the square of the maximal ideal of the regular local $A[[u, v]]$).

REMARK 26.2.2. It is a very special feature of the regularity conditions in our setting (including for the D_i 's) that the abstract $a \in A$ has an $A^\times$ -multiple which comes from $\mathcal{O}_{S,s}$ (no completion). More specifically, for a regular local ring C with dimension ≥ 2 and nonzero non-unit $f \in \hat{C}$

typically there is no $\widehat{C}^\times$ -multiple of f that lies in C . In geometric terms (since regular local rings are UFD's, so all height-1 primes are principal), $\text{Spec}(\widehat{C}) \rightarrow \text{Spec}(C)$ is usually *not* bijective between their sets of codimension-1 points. Put yet another way, this says that some nonzero element of $\widehat{C}$ remains a non-unit after inverting all nonzero elements of C , or equivalently in terms of $K := \text{Frac}(C)$ that $\widehat{C} \otimes_C K$ is *not* a field.

To give an example, the idea is to use a “transcendental curve” in $\mathbf{C}^2$ through the origin, which should have no “algebraic branches”. More precisely, and in a way that works in any characteristic, consider $C = R[x]_{(\pi, x)}$ for a non-complete dvr R with uniformizer π and fraction field F , and let $f = x - h \in \widehat{R}[[x]] = \widehat{C}$ for $h \in \pi\widehat{R} - R$ to be determined. (You can take $R = k[t]_{(t)}$ for a field k , or $R = \mathbf{Z}_{(p)}$.) It suffices to rule out that $f|p(x)$ in $\widehat{C}$ for some nonzero $p \in R[x]$ (even stronger than ruling out that some element of C is a $C^\times$ -multiple of f).

Suppose to the contrary, so some nonzero $p \in R[x]$ is equal to fg for a nonzero $g \in \widehat{R}[[x]]$. The $\widehat{R}$ -algebra evaluation homomorphism $\widehat{R}[[x]] \rightarrow \widehat{R}$ defined by $x \mapsto h$ (meaningful since $h \in \pi\widehat{R}$) carries f to 0, so $p(h) = 0$. This makes h algebraic over F , so as long as $\widehat{F}$ is non-algebraic over F we can pick any $\xi \in \widehat{F} - F$ and use $h = \pi^N \xi$ for sufficiently big N . It remains to make examples of a field F that is *non-complete* for a nontrivial discrete valuation such that its completion $\widehat{F}$ is non-algebraic over F . For F countable this always holds for cardinality reasons (i.e., $\widehat{F}$ is uncountable due to series expansions but an algebraic closure of F is countable), so that suffices to making examples.

Beware that it *can* happen for some non-complete excellent dvr R that its fraction field F has $\widehat{F}$ algebraic over F . For example, let $\{x_i\}$ be a (certainly non-empty) transcendence basis for $\mathbf{Q}_p$ over $\mathbf{Q}$ and let $F = \mathbf{Q}(\{x_i\})$ equipped with the restriction v of the p -adic valuation on $\mathbf{Q}_p$. Note that $\mathbf{Q}_p/F$ is a nontrivial extension since F has (infinitely many) nontrivial automorphisms via the non-empty transcendence basis whereas $\mathbf{Q}_p$ has no nontrivial automorphisms as an abstract field. The valuation ring R of v on F is a discrete valuation ring, so $F = \text{Frac}(R)$ and R is excellent (since $\text{char}(F) = 0$). The inclusion of F into $\mathbf{Q}_p$ is dense, so this identifies $\mathbf{Q}_p \neq F$ with $\widehat{F}$. By design, the non-trivial extension $\widehat{F}/F$ is thereby algebraic.

Lecture 27

Geometry for codimension-2 singularities

§27.1 Description after completion

Taking the same setup as §26.1, made more specific as in the discussion preceding Proposition 26.2.1, so $S = \operatorname{Spec}(A)$ for a regular complete local noetherian ring $(A, \mathfrak{m})$ and $x \in X$ is a non-smooth rational point in the closed fiber. Let $\kappa = A/\mathfrak{m}$. In §26.2 we discussed that $\hat{\mathcal{O}}_{X,x} \simeq A[[u, v]]/(Q - a)$ as A -algebras for $Q(u, v) \in A[[u, v]]$ a residually non-degenerate quadratic form over A and $a \in \mathfrak{m}$ that we will describe later. Recall from Appendix D that we may replace Q with any other quadratic form over A having the same reduction (as all lifts of the same non-degenerate binary quadratic form over κ are isometric), such as a lift over the algebraic localization $\mathcal{O}_{S,s} \subset A$ when $A = \hat{\mathcal{O}}_{S,s}$ for a regular excellent connected noetherian (S, s) .

PROPOSITION 27.1.1. We have $|\operatorname{Spec}(A/(a))| \subset D$.

We saw in the discussion after Proposition 26.2.1 that if this holds then $a \in A^\times \cdot \prod_i t_i^{n_i}$ with $n_i \geq 0$ (and some $n_i > 0$); clearly $\sum n_i \geq 2$ if and only if $x \in \operatorname{sing}(X)$. Multiplying Q and a by a common element in $A^\times$ would then give us that

$$\hat{\mathcal{O}}_{X,x} \simeq A[[u, v]]/(Q - \prod_i t_i^{n_i}).$$

Since residually even-rank non-degenerate quadratic spaces over A are isomorphic when they are residually isomorphic, we could then make a linear change of variables in u and v to ensure that $Q \in \mathcal{O}_s[u, v]$ when $A = \hat{\mathcal{O}}_{S,s}$ for a regular excellent connected noetherian (S, s) .

Proof of Proposition 27.1.1. By Artin approximation, there exists a common étale neighborhood (Y, y) of (X, x) and $(\operatorname{Spec}(A[u, v]/(Q - a)), (s, 0, 0))$:

$$\begin{array}{ccccc} & & (Y, y) & & \\ & \swarrow q' & & \searrow q & \\ (X, x) & & & & (\operatorname{Spec}(A[u, v]/(Q - a)), (s, 0, 0)) \\ & \searrow f & & \swarrow h & \\ & & (\operatorname{Spec} A, s) & & \end{array}$$

with q and q' étale. Since f is smooth over $S - D$, to prove the claim it is enough to show that all fibers of f over $\operatorname{Spec}(A/(a))$ are *not* smooth. We have a section $\sigma : \operatorname{Spec}(A/(a)) \rightarrow \operatorname{Spec}(A[u, v]/(Q - a))$ via $u = v = 0$, sending s to $(s, 0, 0)$. This section is a closed subscheme that

is *local* and contained in $\text{sing}(h)$, fitting into the diagram

$$\begin{array}{ccccc}
 & (Y, y) & & & \\
 q' \swarrow & & q \searrow & & \\
 (X, x) & & (\text{Spec}(A[u, v]/(Q - a)), (s, 0, 0)) & & \\
 f \searrow & & \swarrow h & & \uparrow \sigma \\
 & (\text{Spec } A, s) & & & (\text{Spec}(A/(a)), s) \\
 & \nwarrow \text{closed} & & &
 \end{array}$$

For $Z(a) := \text{Spec}(A/a) \subset \text{Spec}(A)$, the image $q(Y)$ is open in $\text{Spec}(A[u, v]/(Q - a))$ by étaleness of q and it meets $\sigma(Z(a))$ (at $(s, 0, 0) = q(y)$), so $q(Y) \supset \sigma(Z(a))$ by locality of $Z(a)$. Hence, $q^{-1}(\sigma(Z(a))) \twoheadrightarrow \sigma(Z(a))$ under q .

By étaleness of q and the containment $\sigma(Z(a)) \subset \text{sing}(h)$, we then have that $q^{-1}(\sigma(Z(a))) \subset \text{sing}(h \circ q) = \text{sing}(f \circ q')$. Since q' is étale, q' carries $\text{sing}(f \circ q')$ into $\text{sing}(f)$. Thus, the subset $q'(q^{-1}(\sigma(Z(a)))) \subset X$ consists entirely of non-smooth points over $\text{Spec } A$ and it maps onto $Z(a)$. This ensures $Z(a)$ is disjoint from the open complement of D (as $X \rightarrow S = \text{Spec}(A)$ is smooth over $S - D$), which is to say $|Z(a)| \subset |D|$ as desired. $\square$

§27.2 Consequences prior to completion

Assume we are in the setup of 26.1.1, and let $T \subset \text{sing}(X) (\subset \text{sing}(X/S) \subset f^{-1}(D))$ be an irreducible component of $\text{sing}(X)$ of codimension 2 in X . Give T the reduced structure. In the diagram

$$\begin{array}{ccc}
 T & \longrightarrow & X \\
 \downarrow & & \downarrow f \\
 D & \xrightarrow{\text{sncd}} & S
 \end{array}$$

we have by flatness of f that $2 = \dim \mathcal{O}_{X, \eta_T} = 1 + \dim \mathcal{O}_{S, f(\eta_T)}$, so $\dim \mathcal{O}_{S, f(\eta_T)} = 1$. But $f(\eta_T) \in D$ and D has pure codimension-1 in S , so $f(\eta_T)$ is the generic point of some irreducible component of D , say D_{i_T} . The quasi-finite proper map $T \rightarrow D_{i_T}$ is then finite and surjective. Recall that every D_i is regular since D is an sncd.

PROPOSITION 27.2.1. The map $T \twoheadrightarrow D_{i_T}$ is étale, so T is regular and hence $\hat{\mathcal{O}}_{T, x}$ is regular (thus a domain!) for all $x \in T$.

Proof. Let $Y := \text{sing}(X/S)_{\text{red}} \subset f^{-1}(D)$ be the reduced closed subscheme underlying the non-smooth locus over S , so $T \subset Y$ as closed subschemes of X . Topologically, Y is cut out by the annihilator ideal of $\Omega_{X/S}^2$ and so Y as a scheme is defined by the radical of that annihilator.

We claim that $\Omega_{T/D_{i_T}}^1 = 0$ (by Nakayama's Lemma this is the same as $T \rightarrow D_{i_T}$ having étale fibers). To prove this, it suffices to show $\Omega_{Y/D}^1 = 0$ since $\Omega_{T/D_{i_T}}^1 = \Omega_{T/D}^1$ is a quotient of $\Omega_{Y/D}^1$. This vanishing property is étale-local on Y and D , so the properness of $X \rightarrow S$ is not relevant and hence we temporarily drop it (just assume $X \rightarrow S$ is an open semistable curve with regular excellent noetherian connected S).

Since Proposition 26.2.1 has now been proved, we are reduced to checking the “model case” $S = \operatorname{Spec}(A)$ for a regular local noetherian ring A , and $X = \operatorname{Spec}(A[u, v]/(Q - \prod t_i^{n_i}))$ and $D = \operatorname{Spec}(A/(\prod t_i))$ for $Q \in A[u, v]$ a residually non-degenerate quadratic form and $\{t_1, t_2, \dots\}$ part of a regular system of parameters of A . But then Y is cut out in X by (u, v) , so $Y \rightarrow D$ is the natural map $\operatorname{Spec}(A/(\prod_{n_i > 0} t_i)) \rightarrow \operatorname{Spec}(A/(\prod t_i))$ that is a closed immersion. (Note we are working étale-locally on the original situation, so this is *not* proving $Y \rightarrow D$ is a closed immersion in the original situation.) This completes the proof that $\Omega_{T/D_{i_T}}^1 = 0$.

Over any non-empty affine open $\operatorname{Spec} B \subset D_{i_T}$, the map $T \rightarrow D_{i_T}$ becomes a finite surjection $\operatorname{Spec} C \rightarrow \operatorname{Spec} B$ where B is a regular (hence normal) domain, C is a domain (so $B \rightarrow C$ is injective), and $\Omega_{C/B}^1 = 0$. Now a small miracle happens: by [FK88, Lemma 1.5], for any module-finite ring extension $B \hookrightarrow C$ of domains with B normal noetherian, if $\Omega_{C/B}^1 = 0$ then $B \rightarrow C$ is étale (in particular, it is flat). $\square$

For any $x \in \operatorname{sing}(X/S)$ we have a commutative diagram

$$\begin{array}{ccc}
 & (Y, y) & \\
 \swarrow & & \searrow \\
 (X, x) & & (\operatorname{Spec}(A[u, v]/(Q - \prod_i t_i^{n_i(x)}), (s', 0, 0)) \\
 \searrow & & \swarrow \\
 & (S, s) &
 \end{array} \tag{27.2.1}$$

where $(\operatorname{Spec} A, s') \rightarrow (S, s)$ is an étale neighborhood realizing the finite separable residue field extension $k(x)/k(s)$ and $Q(u, v)$ is a *fiberwise* non-degenerate binary quadratic form over A . The tuple of exponents $n_i(x)$ is uniquely determined (independently of the choices of étale neighborhoods) due to the uniqueness of a up to a unit multiple in Theorem 22.2.4.

PROPOSITION 27.2.2. Let $T \subset \operatorname{sing}(X)$ be an irreducible component with codimension 2 in X . Let η_T be the generic point of T , and η_i the generic point of D_i , so $f(\eta_T) = \eta_{i_T}$ and $R := \mathcal{O}_{S, \eta_{i_T}}$ is a dvr.

For every $x \in T$, $n_{i_T}(x)$ equals the measure of irregularity at the non-regular point η_T in the special fiber of the localization X_R over R . In particular, $n_{i_T}(x) \geq 2$ and it is independent of x .

Proof. In the diagram (27.2.1), since the étale map $Y \rightarrow X$ is open and hits $x \in T$, it also hits η_T ;

pick $y' \in Y$ over η_T . Localizing throughout at the codimension-1 point $\eta_{i_T} \in S$ turns all t_i for $i \neq i_T$ into units and for the dvr $R := \mathcal{O}_{S, \eta_{i_T}}$ provides a common étale neighborhood of (X_R, η_T) and a point ξ (image of y') in the special fiber of $\text{Spec}(R'[u, v]/(Q' - t_{i_T}^{n_{i_T}(x)}))$ over a local-étale extension R' of R , where Q' is a residually trivial binary quadratic form (an $R'^\times$ -multiple of Q over A).

The point ξ is non-regular since the same holds for η_T (as $T \subset \text{sing}(X)$), so necessarily $n_{i_T}(x) > 0$, and ξ must be the unique singularity in the special fiber $Q' = 0$ over the residue field of R' . It follows that $n_{i_T}(x)$ is the measure of irregularity at ξ for the open semistable curve $Q' = t_{i_T}^{n_{i_T}(x)}$ over R' , so it is also the measure of irregularity at η_T for X_R over R . The latter is certainly independent of x and is at least 2 since η_T is not a regular point. $\square$

Lecture 28

Semistability of a blow-up

§28.1 Passage to a model case

Recall we have the following setup:

$$\begin{array}{ccccc} T & \subseteq & \text{sing}(X) & \subseteq & X \\ \downarrow & & & & \downarrow f \\ D_{i_T} & \subseteq & D & \subseteq_{\text{sncd}} & Y \end{array}$$

where

- the map $f : X \rightarrow Y$ is a proper semistable curve with (geometrically) connected fibers, where Y is a smooth projective variety of dimension $d - 1 \geq 1$ over an algebraically closed field k (so X is normal);
- the closed set $D \subset Y$ is an sncd, and f is smooth over $Y - D$;
- the set of non-regular points $\text{sing}(X) \subset X$ (with codimension ≥ 2 by normality) has an irreducible component T of codimension 2 in X ;
- the irreducible component T maps finitely onto an irreducible (and regular, by the sncd condition) component D_{i_T} of D , with $T \rightarrow D_{i_T}$ also étale, so T is regular.

In Proposition 27.2.2 we saw that if η_T, η_{i_T} denote the generic points of T, D_{i_T} respectively then for the dvr $R = \mathcal{O}_{Y, \eta_{i_T}}$ the measure of irregularity $n_T \geq 2$ of the localization X_R is precisely the exponent $n_{i_T}(x)$ of t_{i_T} in the étale-local description of (X, x) (see the commutative diagram in (27.2.1)).

With the setup as described above, we hope for two things:

- (1) The blow-up $X' := \text{Bl}_T(X)$ is also semistable over Y (it is smooth over $Y - D$ since $X' \rightarrow X$ is an isomorphism over $X - T$).
- (2) The blowup X' is “better” than X in terms of singularities: either it has fewer codimension-2 irreducible components of the non-regular locus, or some such irreducible component has a lower measure of irregularity.

Ultimately we will see that there exists at most one codimension-2 irreducible component $T' \subset \text{sing}(X')$ lying over T , and when such a T' exists we will have $n_{T'} = n_T - 2$. Thus (since there are only finitely many such T , as $\text{sing}(X)$ has finitely many irreducible components), after finitely many such blow-ups we will reduce to the case where $\text{codim}_X(\text{sing}(X)) \geq 3$, which we will grapple with afterwards.

REMARK 28.1.1. We also need to (eventually) deal with the structure of our closed set $Z = f^{-1}(D) \cup (\cup_j \tau_j(Y)) \subset X$ so that it ends up having the desired properties as stated in Theorem 1.2.4 (as we need for success of the induction: recall how we arranged D to be an sncd in Y , a property that in turn underlies all of our étale-local study of singularities on X). We will address this matter after resolving the singularities on X .

To check semi-stability of $\mathrm{Bl}_T(X) \rightarrow Y$, our task is Zariski-local over X near points of T (since $\mathrm{Bl}_T(X) \rightarrow X$ is an isomorphism over $X - T$), so we aim to prove it is open semistable near the points of T . It suffices to check this étale-locally at points $x \in T \subset X$ (recall that “open semistable” amounts to flatness and a condition on geometric fibers over Y). Let $(\mathrm{Spec}(R), f(x))$ be a (pointed) affine neighborhood of $(Y, f(x))$. By the structure of ordinary double point singularities via Artin approximation (see (27.2.1)), there exists a common residually trivial étale neighborhood (W, w) of (X, x) and $\mathrm{Spec}(A[u, v]/(Q - a), (y', 0, 0))$ over $(\mathrm{Spec}(R), f(x))$ where $(\mathrm{Spec}(A), y')$ is a pointed étale neighborhood of $(\mathrm{Spec}(R), f(x))$ realizing the residue field extension $k(x)/k(f(x))$ and $Q(u, v)$ is a residually non-degenerate quadratic form. For actual calculation purposes we need a more explicit quadratic form. By passing to yet another pointed étale neighborhood of $(\mathrm{Spec}(A), y')$ to make the residue field big enough to residually split Q , we can replace Q over A with uv over R .

In other words, we have the following diagram of (not necessarily residually trivial!) pointed étale neighborhoods:

$$\begin{array}{ccc}
 & (W, w) & \\
 q_1 \swarrow & & \searrow q_2 \\
 (X, x) & & (\mathrm{Spec}(R[u, v]/(uv - \prod_i t_i^{n_i(x)}), (f(x), 0, 0)) \\
 \searrow & & \swarrow \\
 & (Y, f(x)) &
 \end{array} \tag{28.1.1}$$

Informally, our aim is to carefully analyze the “zig-zag” diagram of étale neighborhoods to justify reducing our problem for X near x to an analogue in the “model example” on the right side of the diagram. This is a version of the higher-dimensional base Y of what we did in our work with semistable curves over a dvr, and such zig-zag work with careful attention to irreducible components is part of the technical justification for being able to treat étale neighborhoods “as if” they’re like neighborhoods in analytic geometry (which they’re not, logically speaking).

Our aim is to show $\mathrm{Bl}_T(X)$ is open semistable over Y at all points over the arbitrary choice of $x \in T$. We may therefore replace X with the open neighborhood $q_1(W)$ of $q_1(w) = x$ (here we use openness of the étale q_1 , and recall we don’t care about properness of X for the time being), and likewise replace T with its irreducible open subset $q_1(X) \cap T \ni x$, to ensure $q_1 : W \twoheadrightarrow X$. By étaleness of q_1 we have $\mathrm{sing}(W) = q_1^{-1}(\mathrm{sing}(X))$. Moreover, $q_1^{-1}(T)$ is of pure codimension-2 in W , by flatness and quasi-finiteness of q_1 , with the étale map $q_1^{-1}(T) \rightarrow T$ surjective since we arranged

$q_1 : W \rightarrow X$ to be surjective.

We claim W can be shrunk further around w to arrange that $q_1^{-1}(T)$ is irreducible. Since q_1 is étale, regularity of T is preserved under taking its preimage, and so there must exist a unique irreducible component of $q_1^{-1}(T)$ in which w lies. In particular the irreducible components of $q_1^{-1}(T)$ are its connected components, so we can shrink W some more around w if need be to remove the other connected components of $q^{-1}(T)$ (and then replacing X with the new $q_1(W)$). In this way we arrange that $\mathcal{T} := q^{-1}(T)$ is irreducible.

Since the formation of blow-ups commutes with flat (e.g., étale) base change, we see that in the commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_{\mathcal{T}}(W) & \longrightarrow & W \\ \downarrow & & \downarrow \\ \mathrm{Bl}_T(X) & \longrightarrow & X \longrightarrow Y \end{array}$$

the square is cartesian with the left vertical arrow an étale *surjection* since the right vertical arrow is. Hence, the semistability of $\mathrm{Bl}_T(X) \rightarrow Y$ is reduced to the case of $\mathrm{Bl}_{\mathcal{T}}(W) \rightarrow Y$. This completes half of the work with chasing through the zig-zag of étale neighborhoods in (28.1.1).

Next, setting

$$B := R[u, v]/(uv - \prod_i t_i^{n_i(x)})$$

for convenient notational shorthand, we observe that $\mathrm{sing}(B) \subset \mathrm{Spec}(B)$ is closed of codimension ≥ 2 (as $\mathrm{Spec}(B)$ is open semistable over $\mathrm{Spec}(R)$ and smooth over $\mathrm{Spec}(R) \cap (Y - D)$, so B is normal). Moreover, étaleness of $q_2 : W \rightarrow \mathrm{Spec}(B)$ tells us $\mathrm{sing}(W) = q_2^{-1}(\mathrm{sing}(B))$, and thus for dimension reasons $q_2(\mathcal{T})$ has to dominate some irreducible component $T' \subset \mathrm{sing}(B)$ of codimension 2 in $\mathrm{Spec}(B)$. This T' is regular for exactly the same reason that T is regular: it is étale over an irreducible component of the sncd $D \subset Y$ (the proof of this for $T \rightarrow D$ did not use properness of $X \rightarrow Y$, only that it is open semistable and smooth over $Y - D$; finiteness of $T \rightarrow D$ was never needed for this, just quasi-finiteness).

Thus, the closed subset $q_2^{-1}(T') \subset \mathrm{sing}(W)$ is regular (as it is étale over T') and of pure codimension 2 in W (for dimension reasons since q_2 is quasi-finite flat). so it is a union of irreducible components of $\mathrm{sing}(W)$. But $q_2^{-1}(T')$ contains $\mathcal{T}$ by design, so $\mathcal{T}$ is one of the irreducible components of $q_2^{-1}(T')$. The irreducible components of $q_2^{-1}(T')$ are its connected components by regularity, so $\mathcal{T}$ is the connected component of $q_2^{-1}(T')$ through w . By shrinking W some more around w and X around x (to keep $W \rightarrow X$ surjective), we can ensure $q_2^{-1}(T') = \mathcal{T}$. Hence, we get the following *cartesian* square:

$$\begin{array}{ccccc} \mathrm{Bl}_{\mathcal{T}}(W) & \longrightarrow & \mathrm{Bl}_{T'}(\mathrm{Spec}(B)) & & \\ \downarrow & & \downarrow & & \\ (W, w) & \longrightarrow & \mathrm{Spec}(B) & \longrightarrow & Y \end{array}$$

To prove $\text{Bl}_{\mathcal{T}}(W)$ is open semistable over Y , it therefore suffices to check open semistability of $\text{Bl}_{T'}(\text{Spec}(B)) \rightarrow Y$. It remains to identify T' . We'll show it is cut out by the ideal (u, v, t_i) for some i (which is prime, as $B/(u, v, t_i) = R$ is a domain).

As an irreducible component of the Y -quasi-finite $\text{sing}(\text{Spec}(B))$ having codimension 2 in $\text{Spec}(B)$ (i.e., of dimension $d - 2$), T' dominates an irreducible closed subset of the sncd $\text{Spec}(R) \cap D$ in the $(d - 1)$ -dimensional regular $\text{Spec}(R)$. But $\text{Spec}(R) \cap D = \text{Spec}(R/(\prod t_i))$ has pure dimension $(d - 1) - 1 = d - 2$ too, so T' dominates an irreducible component of $\text{Spec}(R) \cap D$. The local ring $R_{f(x)}$ is regular and hence a domain in which the t_i 's are part of a regular system of parameters, so each $R_{f(x)}/(t_i)$ is regular and thus a *domain*. By spreading-out considerations, if we replace $\text{Spec}(R)$ with a suitable affine open around $f(x)$ (and then shrink everything over $\text{Spec}(R)$ likewise) then each $R/(t_i)$ is also a domain (reduced and irreducible spectrum), so T' dominates some $\text{Spec}(R/(t_i))$.

We have shown T' is contained in $\text{Spec}(B/(t_i)) = \text{Spec}(R[u, v]/(uv))$, yet it also lies in the zero locus of (u, v) in $\text{Spec}(B)$ since fiberwise that is the locus of singularities over $\text{Spec}(R)$. In other words, T' is contained in $\text{Spec}(B/(u, v, t_i)) = \text{Spec}(R)$ which is itself integral of dimension $d - 2$, so $T' = \text{Spec}(B/(u, v, t_i))$ as desired.

Finally, we claim $i = i_T$. Since $q_1^{-1}(T) = \mathcal{T} = q_2^{-1}(T')$, we see that T and T' have the *same* image-closure in $\text{Spec}(R) \cap D$, which is $\text{Spec}(R) \cap D_{i_T}$. This forces $i = i_T$, so we have finally reduced our task to a “model example”: prove open semistability over R for the blow-up along (u, v, t_{i_T}) of B with $n_{i_T}(x) = n_T \geq 2$. Since our original task could focus on $\text{Bl}_T(X)$ at points over the chosen arbitrary $x \in X$, we can replace R with its (positive-dimensional) localization at the *non-generic* point $f(x)$ to make R *regular local*.

§28.2 Explicit blow-up calculation

Let R be a regular local ring of positive dimension δ , $\{t_j\}$ part of a regular system of parameters, and $\{n_j\}$ positive integers such that $n_1 \geq 2$. Define $B = R[u, v]/(uv - \prod_{j=1}^r t_j^{n_j})$, and $I = (u, v, t_1)$ (so $B/I = R/t_1$ is regular local of dimension $\delta - 1$). Our aim is to show that $\text{Bl}_I(B)$ is an open semistable R -curve. The case $\delta = 1$ has been done already (our earlier work over discrete valuation rings), and we now wish to carry out such a calculation for general $\delta > 0$. Our only aim is to prove “open semistable” and not to keep track of any irreducible components in the non-regular locus or any function fields of such components, so the problem is solely one of examining the blow-up charts $D_+(u), D_+(v), D_+(t_1)$ to check that each of these affine open subschemes of the blow-up is an open semistable R -curve. Since u and v play entirely symmetric roles in the setup, the task is really just to study $D_+(u)$ and $D_+(t_1)$.

For $D_+(u)$, we introduce new variables v' and t'_1 subject to the relations $v = v'u$ and $t_1 = t'_1 u$. We need to determine the quotient of the R -algebra

$$R[u, v, v', t'_1]/(uv - \prod t_j^{n_j}, v - v'u, t_1 - t'_1 u) = R[u, v', t'_1](u^2 v' = (t'_1)^{n_1} u^{n_1} t_2^{n_2} \cdots t_r^{n_r}, t_1 - t'_1 u)$$

modulo its u^∞ -torsion. Since $n_1 \geq 2$ we can cancel the factor of u^2 in the final relation to obtain the relation

$$v' = (t'_1)^{n_1} u^{n_1-2} t_2^{n_2} \cdots t_r^{n_r}$$

that permits us to remove v' . That leaves us with the quotient

$$R[u, v', t'_1]/(t_1 - t'_1 u)$$

which is visibly a regular domain (in particular, vanishing u^∞ -torsion) and R -semistable.

Next, for $D_+(t)$ we introduce variables u'', v'' and relations $u = u''t_1, v = v''t_1$. This amounts to finding the quotient of the R -algebra

$$R[u, v, u'', v'']/(uv - \prod t_j^{n_j}, u - u''t_1, v - v''t_1) = R[u'', v'']/(u''v''t_1^2 - \prod t_j^{n_j})$$

modulo its t_1^∞ -torsion. Since $n_1 \geq 2$ we can cancel a factor of t_1^2 from the final relation, leaving us with

$$R[u'', v'']/(u''v'' - t_1^{n_1-2} \prod_{j=2}^r t_j^{n_j}).$$

This is visibly R -semistable.

Lecture 29

Irreducible components of $\text{sing}(X)$

§29.1 Geometry of irreducible components

Take the setup as in §28.1, but assume instead that $\text{codim}_X(\text{sing}(X)) \geq 3$ and $\text{sing}(X)$ is not empty. For any $x \in \text{sing}(X)$, as in §28.1 consider $(\text{Spec } R, f(x))$ a sufficiently small (pointed) affine open neighborhood of $(Y, f(x))$ that meets only those D_i 's which pass through $f(x)$ and for which the invertible $\mathcal{I}_{D_i}|_{\text{Spec}(R)}$ is free with generator t_i (so $\{t_i\}$ is part of a regular system of parameters in $R_{f(x)}$). By Artin Approximation we have

$$\begin{array}{ccc}
 & (W, w) & \\
 q_1 \swarrow & & \searrow q_2 \\
 (X, x) & & (\text{Spec } B, \xi) \\
 \searrow & & \downarrow \\
 & (Y, f(x)) \xleftarrow{\text{open}} (\text{Spec } R, f(x)) &
 \end{array} \tag{29.1.1}$$

with q_1 and q_2 étale, where $B := R[u, v]/(uv - \prod_i t_i^{n_i(x)})$ and $\xi := (f(x), 0, 0)$.

The non-regularity of the completion $\widehat{B}_\xi = \widehat{R_{f(x)}}[u, v]/(uv - \prod_i t_i^{n_i(x)})$ implies $\sum_i n_i(x) \geq 2$. By shrinking R enough around $f(x)$ we can ensure B is a domain, since by Gauss' Lemma $B_{f(x)} = R_{f(x)}[u, v]/(uv - \prod_i t_i^{n_i(x)})$ is a domain (as $R_{f(x)}[u, v]$ inherits the UFD property from the regular local $R_{f(x)}$).

PROPOSITION 29.1.1. If $n_i(x) \neq 0$ then $n_i(x) = 1$.

Proof. Note that $B/(u, v, t_i) = R/(t_i)$ for any i with $n_i(x) > 0$. Its localization at $f(x)$ is $R_{f(x)}/(t_i)$ that is regular local and hence a domain, so by shrinking R around $f(x)$ we can ensure each $R/(t_i)$ is a domain too. (Beware that the UFD property of $R_{f(x)}$ doesn't generally "spread out" to an affine Zariski-open neighborhood of $f(x)$, as already happens for many Dedekind domains with infinite class group. Fortunately we have not needed that to spread out the "domain" property from a local ring of a noetherian ring at a prime.)

Suppose $n_{i_0} := n_{i_0}(x) \geq 2$ for some i_0 , so $T' := \text{Spec}(R/(t_{i_0})) = Z(u, v, t_{i_0}) \subset \text{Spec } B$ is integral and it has codimension 2 near ξ in $\text{Spec } B$ since $\dim B_\xi = 1 + \dim R_{f(x)} = 2 + \dim(R_{f(x)}/(t_{i_0}))$. For η_{i_0} the generic point of D_{i_0} , $A := R_{\eta_{i_0}}$ is a dvr. Note then that $\text{Spec}(B_{\eta_{i_0}}) \rightarrow \text{Spec } A$ is an open semistable curve over a dvr and $\eta_{T'}$ is a nonsmooth point in the special fiber with measure of irregularity $n_{i_0}(x) \geq 2$. This implies that $(B_{\eta_{i_0}})_{\eta_{T'}}$ is not regular, so $\eta_{T'} \in \text{sing}(B)$ and hence $T' \subset \text{sing}(B)$ (as the singular locus is a closed subset of $\text{Spec}(B)$).

By normality of the domain B (as it is an open semistable curve over R with generic fiber

smooth), its singular locus has codimension at least 2. But T' in $\text{sing}(B)$ has codimension 2 in $\text{Spec}(B)$, so it must be an irreducible component of $\text{sing}(B)$ and $\text{codim}_B(\text{sing}(B)) = 2$. But $q_2(W)$ is an open subset of $\text{Spec } B$ that contains $q_2(w) = \xi \in T'$, so $q_2^{-1}(T') \subset W$ is *non-empty* and contained in $\text{sing}(W)$ since q_2 is étale and $T \subset \text{sing}(X)$. Yet q_2 is quasi-finite and flat, and thus $\text{codim}_{q_2^{-1}(T)}(W) = 2$, which implies that $\text{sing}(W)$ has codimension 2 inside W (its codimension is at least 2 since W inherits normality from X by étaleness of q_1). For an irreducible component $\mathcal{T}$ of $\text{sing}(W)$ of codimension 2 in W , we have

$$\begin{array}{ccccc} W & \longleftarrow & \text{sing}(W) & \longleftarrow & \mathcal{T} \\ q_1 \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & \text{sing}(X) & \longleftarrow & \overline{q_1(\mathcal{T})} \end{array}$$

with q_1 étale. Therefore, $\overline{q_1(\mathcal{T})}$ has the same codimension in X as $\mathcal{T}$ does in W , which is 2. But $q_1(\mathcal{T}) \subset \text{sing}(X)$ since q_1 is étale and $\mathcal{T} \subset \text{sing}(W)$, so its closure is also contained in the closed $\text{sing}(X) \subset X$. We then have a contradiction as $\text{sing}(X)$ has codimension 3 in X . $\square$

Let $x \in X$, and relabel the D_i so that $f(x) \in D_i$ for exactly $1 \leq i \leq \mu \leq n$. In other words, (X, x) has common local-étale neighborhood with $(\text{Spec } B, \xi)$ for $B = R[u, v]/(uv - \prod_{i=1}^{\mu} t_i)$ and $\xi = (f(x), 0, 0)$. Since $B_{\xi} = R_{f(x)}[u, v]_{\xi}/(uv - \prod_{i=1}^{\mu} t_i)$ inherits non-regularity from $\mathcal{O}_{X, x}$, as in the preceding proof the sum $\sum_i n_i(x) = \mu$ is at least 2. The t_i 's that occur have the corresponding D_i containing $f(x)$, so all points in $\text{sing}(X)$ lie over at least two distinct D_i 's. Thus, by shrinking Y around $f(x)$ we can arrange that $\text{sing}(X)$ is covered by the closed subsets $\text{sing}(X) \cap f^{-1}(D_i \cap D_j)$ for all pairs $1 \leq i < j \leq \mu$.

- PROPOSITION 29.1.2.** 1. For $1 \leq i_0 < j_0 \leq \mu$, $\text{sing}(X) \cap f^{-1}(D_{i_0} \cap D_{j_0})$ with the reduced structure is regular, so its unique connected (= irreducible!) component E_{ij} through x has codimension 3 and the map $f : E_{ij} \rightarrow D_i \cap D_j$ is étale (note that $D_i \cap D_j$ is regular of pure codimension 2 as D is an sncd in the regular Y).
2. If $E_{\alpha}, E_{\beta} \subset \text{sing}(X)$ are distinct irreducible components equipped with the reduced structure and $E_{\alpha} \cap E_{\beta} \neq \emptyset$ then the schematic intersection $E_{\alpha} \cap E_{\beta}$ is regular with codimension 4 or 5 near each of its points.

Proof. First we prove (1). The approach is the usual zig-zag through a common étale neighborhood to pass to a model example with which we can make explicit calculations. Let $(\text{Spec } R, f(x))$ be an open affine neighborhood of $f(x)$ in Y as in (29.1.1) with $B = R[u, v]/(uv - \prod_{i=1}^{\mu} t_i)$ a domain (this may require making $\text{Spec}(R)$ sufficiently small around $f(x)$). We can likewise arrange that all $R/(t_i t_j)$ for $1 \leq i < j \leq \mu$ are domains. The fibral non-smooth locus (which contains $\text{sing}(X)$) consists of $(u, v) = (0, 0)$ over zeros of $\prod_{i=1}^{\mu} t_i$ in $\text{Spec}(R)$, so near $\xi := (f(x), 0, 0)$, for $1 \leq i < j \leq \mu$ we see that $f^{-1}(D_i \cap D_j)$ coincides with $Z(u, v, t_i, t_j) \simeq \text{Spec}(R/(t_i t_j)) = D_i \cap D_j$

(integral by inspection) that has local ring at ξ equal to $R_{f(x)}/(t_i t_j)$ which is non-regular. Thus, $\bigcup_{i < j} Z(u, v, t_i, t_j) \subset \text{sing}(B)$. But $\text{sing}(B) \subset \bigcup_{i < j} f^{-1}(D_i \cap D_j) = \bigcup_{i < j} Z(u, v, t_i, t_j)$, so they are equal with irreducible components $Z(u, v, t_i, t_j)$. This proves (1) by the zig-zag method through étale neighborhoods (the étaleness of $E_{ij} \rightarrow D_i \cap D_j$ in the general setting is inferred from the isomorphism $Z(u, v, t_i, t_j) \simeq D_i \cap D_j$ in the model example).

For (2), suppose there is a point $x \in E_\alpha \cap E_\beta$, so upon shrinking Y to a small enough affine open $\text{Spec}(R)$ around $f(x)$ as above we have $\alpha = (i, j)$ and $\beta = (i', j')$ with $1 \leq i, j, i', j' \leq mu$ and $i < j, i' < j'$. We again zig-zag through a common étale neighborhood as in (29.1.1) with $B = R[u, v]/(uv - \prod_{l=1}^\mu t_l)$ a domain (again, this may require making $\text{Spec } R$ sufficiently small around $f(x)$). We can express $E_\alpha \cap E_\beta$ as $Z(u, v, t_i, t_j, t_{i'}, t_{j'})$, and note that the local ring at ξ for this scheme-theoretic intersection is

$$R[u, v]_\xi / (u, v, t_i, t_j, t_{i'}, t_{j'}) = R_{f(x)} / (t_i, t_j, t_{i'}, t_{j'})$$

which is regular by inspection. This proves regularity of $E_\alpha \cap E_\beta$.

The dimension of $R_{f(x)}$ is $\dim B_\xi - 1$, and $\{t_i, t_j, t_{i'}, t_{j'}\}$ is a set of at least 3 (and at most 4) elements of a regular system of parameters in $R_{f(x)}$. Therefore the dimension of $R_{f(x)} / (t_i, t_j, t_{i'}, t_{j'})$ (i.e. the codimension of $E_\alpha \cap E_\beta$ near ξ) is either 4 or 5. $\square$

§29.2 Interaction of Z with modifications of X

In the setup in §28.1 we have $Z = f^{-1}(D) \cup (\bigcup_i \tau_i(Y))$ for disjoint sections $\tau_i : Y \rightarrow \text{sm}(X/Y) \subset X^{\text{sm}}$. The τ_i 's are “unaffected” by blow-up in the singular locus, such as the successive blow-ups at irreducible components T of the singular locus with $\text{codim}_X(T) = 2$. Letting $\phi : X_1 \rightarrow X$ be the modification over Y centered over $\text{sing}(X)$ as a composition of blow-ups at such T 's,

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi} & X \\ & \searrow f_1 & \swarrow f \\ & Y & \end{array}$$

we have $Z_1 := \phi^{-1}(Z) = f_1^{-1}(D) \cup (\bigcup_i \tau'_i(Y))$ for $\tau'_i = \phi^{-1} \circ \tau_i$ (ϕ is an isomorphism over X^{sm}) and $\text{codim}_{X_1}(\text{sing}(X_1)) \geq 3$. We can then pass to (X_1, Z_1) , so in this way while keeping track of Z we can arrange that $\text{sing}(X)$ has codimension at least 3 in X . This then puts us in the situation of §29.1, so all irreducible components of $\text{sing}(X)$ (if it is non-empty) have codimension exactly 3. We will analyze the structure of $Z \cap X^{\text{sm}}$ in such circumstances as well as resolve singularities of X to complete the proof next time.

Lecture 30

Analyzing Z and resolving X §30.1 Structure of $Z \cap X^{\text{sm}}$

At the end of §29.2, we reduced to the case (while remaining attentive to Z) $\text{codim}_X(\text{sing}(X)) \geq 3$. More precisely, if $\text{sing}(X) \neq \emptyset$ then all its irreducible components have codimension exactly 3 in X , with each intersection of pairs of such irreducible components (when non-empty) of codimension either 4 or 5 in X . Throughout the blow-ups bringing us to the case $\text{codim}_X(\text{sing}(X)) \geq 3$, X remains a semistable curve over Y . However to resolve these final singularities we will ruin semistability over Y and thus we will need a new axiomatic setup that omits mention of Y .

In the case $d = 2$, we know X is smooth but $Z \subset X$ may not be an sncd and so the case $d = 2$ is still not entirely settled. Our current situation is that $f : X \rightarrow Y$ is a semistable curve that is smooth over $Y - D$ with D an sncd in Y . Moreover, the closed set $Z \subset X$ is described by

$$Z = \left(f^{-1}(D) \cup \left(\bigcup_j \tau_j(Y) \right) \right)_{\text{red}},$$

for pairwise disjoint sections $\tau_j : Y \rightarrow \text{sm}(X/Y) \subset X^{\text{sm}}$. We need to replace these properties with conditions that do not mention Y , starting with:

PROPOSITION 30.1.1. With the above notation and setup, the intersection $Z \cap X^{\text{sm}}$ is a normal crossings divisor in X^{sm} (i.e., some étale cover U' of an open $U \subset X^{\text{sm}}$ containing $Z \cap X^{\text{sm}}$ has the preimage of $Z \cap X^{\text{sm}}$ in U' as an sncd).

In Appendix E, we show for an ncd $D \subset S$ with S regular, there exists a modification $\phi : S' \rightarrow S$ centered over D (i.e., an isomorphism over $S - D$) with regular source such that $\phi^{-1}(D)$ is an sncd in S' . Thus, the proposition settles the case when $d = 2$ or when $X = X^{\text{sm}}$.

Proof. We analyze étale neighborhoods of (X^{sm}, x) for $x \in Z \cap X^{\text{sm}}$ in several cases.

Case 1: $x \notin f^{-1}(D)$

Here $x \in \bigcup_j \tau_j(Y)$ for $\tau_j \in (\text{sm}(X/Y))(Y)$ and the union of closed subsets *disjoint*. Thus x lies in a section τ_{j_0} to the smooth open curve $\text{sm}(X/Y) \rightarrow Y$. Any section to a smooth open curve over a regular base is easily seen to be an sncd, so this case is settled.

Case 2: $x \in f^{-1}(D)$ and $x \notin \bigcup_j \tau_j(Y)$

We separately treat the possibilities that x is a smooth point in $X_{f(x)}$ or not. First suppose $x \in \text{sm}(X/Y)$, so the flat finitely presented $f : X \rightarrow Y$ is smooth near x . Thus, each $f^{-1}(D_i)$ inherits k -smoothness near x from k -smoothness of D_i passing through $f(x)$, and the dimension

formula for flat maps (or expressing an open neighborhood of x in $\text{sm}(X/Y)$) as étale over $\mathbf{A}_Y^1$) then ensures $f^{-1}(D) \subset X$ is an sncd near x since $D \subset Y$ is an sncd near $f(x)$.

Next we may assume $x \notin \text{sm}(X/Y)$; i.e., x is a non-smooth point in the semistable curve $X_{f(x)}$. We have the following étale neighborhoods

$$\begin{array}{ccc}
 & (W, w) & \\
 \swarrow & \downarrow & \\
 (X, x) & & (\text{Spec}(B), \xi = (f(x), 0, 0)) \\
 \downarrow & & \downarrow h \\
 (Y, f(x)) & \supset & (\text{Spec}(R), f(x))
 \end{array}$$

where $\text{Spec}(R)$ is an affine open neighborhood chosen small enough so that $D_i \cap \text{Spec}(R) = \text{Spec}(R/(t_i))$ is a domain (where the invertible $\mathcal{I}_{D_i}|_{\text{Spec}(R)} = t_i R$ is globally principal) and $B = R[u, v]/(uv - t_{i_0})$ for some i_0 (not generally unique) with $f(x) \in D_{i_0}$. We have used that $x \in \text{Reg}(X) = X^{\text{sm}}$ (as this ensures $\sum_i n_i(x) = 1$).

The reason we can arrange each $R/(t_i)$ is a domain by shrinking $\text{Spec}(R)$ around $f(x)$ is that the “domain” property spreads out from stalks of a noetherian ring and $\{t_i\}_i$ recovers part of a regular system of parameters of the regular local ring $R_{f(x)}$ (so each $R_{f(x)}/(t_i)$ is regular and therefore a domain).

Our task is to show $f^{-1}(D)$ is an ncd near x in X . Since the ncd property near a point is checkable using an étale neighborhood, by zig-zagging through the above diagram our task becomes showing the subset $h^{-1}(D) \subset \text{Spec}(B)$ is an sncd near ξ . This is essentially a question about $B_\xi = (R_{f(x)}[u, v])_\xi/(uv - t_{i_0})$. We may and do shrink $\text{Spec}(R) \ni f(x)$ so it only meets those $D_i \ni f(x)$. Now we have a couple of sub-cases to analyze:

- If $i \neq i_0$, then $h^{-1}(D_i)_\xi$ looks like

$$(R_{f(x)}[u, v]_\xi)/(uv - t_{i_0}, t_i) = ((R_{f(x)}/t_i)[u, v]_\xi)/(uv - t_{i_0}),$$

which is a regular local ring, since $\{t_i\}_{j \neq i_0} \cup \{uv - t_{i_0}, u, v\}$ is part of a regular system of parameters for $R[u, v]_\xi = R_{f(x)}[u, v]_\xi$.

- If $i = i_0$ then $h^{-1}(D_{i_0})_\xi$ looks like

$$((R_{f(x)}/t_{i_0})[u, v]_\xi)/(uv),$$

which is the union of $Z(t_{i_0}, u)$ and $Z(t_{i_0}, v)$.

In particular, analyzing either of the descriptions above one can visibly see that taking successive intersections of the $h^{-1}(D_i)$'s yields a regular closed subscheme of the expected codimension for an

sncd, and thus $f^{-1}(D)$ is an ncd, as desired.

Case 3: $x \in f^{-1}(D) \cap \tau_{j_0}(Y)$ for some (unique) j_0

Here we need to show that

$$(f^{-1}(D) \cup \tau_{j_0}(Y))_{\text{red}}$$

near x is an ncd inside X . We'll even show this is an sncd inside X near x .

Note x is in the fibral smooth locus (since it is a point in the image of a section τ_{j_0} to $\text{sm}(X/Y) \rightarrow Y$), and thus $f : X \rightarrow Y$ is smooth near x . Recall the Zariski-local structure of smooth morphisms (an algebro-geometric version of the inverse function theorem) [EGA, IV₄, Cor.17.12.2(d)]: the source is Zariski-locally étale over an affine space over the target. Thus, for a sufficiently small open $U \subset X$ around x we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{q \text{ étale}} & \mathbf{A}_{Y'}^1 \\ \tau_{j_0} \uparrow \downarrow & \swarrow & \\ & Y' & \end{array}$$

where Y' is Y shrunk sufficiently around $f(x)$ so that the image of the section lands in U (note that $\tau_{j_0}(f(x)) = x$ since $x \in \tau_{j_0}(Y)$), and $q \circ \tau_{j_0}$ is the zero-section (as any section of the affine line over a base becomes the zero-section after an additive translation).

Let $D' = D \cap Y'$ (an sncd inside Y') and let $f' : U \rightarrow Y'$ be the smooth structure map (so $f'^{-1}(D') = U \cap f^{-1}(D)$). The inclusion $\tau_{j_0}(Y') \subset q^{-1}(0\text{-section})$ is a clopen immersion since any section to a separated étale map is a clopen immersion, so we obtain the diagram

$$\begin{array}{ccc} (f'^{-1}(D') \cup \tau_{j_0}(Y'))_{\text{red}} & \subset_{\text{clopen}} & q^{-1}((\mathbf{A}_{D'}^1 \cup (\{0\} \times Y'))_{\text{red}}) \subset U \\ & & \downarrow \text{étale} \\ & & (\mathbf{A}_{D'}^1 \cup (\{0\} \times Y'))_{\text{red}} \subset_{\text{sncd}} \mathbf{A}_{Y'}^1 \end{array}$$

expressing $(f^{-1}(D) \cup \tau_{j_0}(Y))_{\text{red}}$ near x as clopen inside the pullback of an sncd along the étale map q . But the pullback of an sncd along an étale map to a regular scheme is again an sncd (check!), so we have the sncd property near x . $\square$

§30.2 New axioms and final blow-ups

We may now give a new axiomatic description of the setup to which we have reduced ourselves. This involves properties of $Z \subset X$ that do not mention Y .

SETUP 30.2.1. We have X a projective variety over an algebraically closed field k with dimension $d \geq 3$ and $Z \subset X$ a reduced closed subscheme such that

- (i) $Z \cap X^{\text{sm}} \subset X^{\text{sm}}$ is an ncd.

- (ii) All irreducible components E of $\text{sing}(X)$ are smooth of codimension exactly 3, and any nonempty intersection of such components is also smooth (though maybe not transverse).
- (iii) For all $x \in (\text{sing}(X) \cap Z)(k)$, we have the following diagram of k -algebra isomorphisms and quotient maps:

$$\begin{array}{ccc}
 \hat{\mathcal{O}}_{X,x} & \xrightarrow{\sim} & k[[u, v, t_1, \dots, t_{d-1}]]/(uv - t_1 \cdots t_s) \\
 \downarrow & & \downarrow \\
 \hat{\mathcal{O}}_{Z,x} & \xrightarrow{\sim} & \hat{\mathcal{O}}_{X,x}/(t_1, \dots, t_r)
 \end{array}$$

with $2 \leq s \leq r \leq d-1$ that may depend on x . (This encodes that $D_1, \dots, D_r$ are exactly the D_i 's passing through $f(x)$, and that Z near x – hence away from the $\tau_j(Y)$'s – is $f^{-1}(D)$ since $x \in \text{sing}(X)$ cannot lie in any of the sections $\tau_j(Y) \subset (\text{sm}(X/Y))(Y)$.)

Axioms (i) and (iii) ensure that the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is invertible by checking all $x \in Z(k)$: use (i) for $x \in X^{\text{sm}}$ and use (iii) for $x \in \text{sing}(X)$. By Artin approximation, (iii) in Setup 30.2.1 provides a model example with which to carry out zig-zag arguments using a common étale neighborhood of (X, x) equipped with $Z \subset X$ for $x \in (\text{sing}(X) \cap Z)(k)$: the pointed scheme $(\text{Spec}(B), 0)$ for

$$B = k[u, v, t_1, \dots, t_{d-1}]/(uv - t_1 \cdots t_s)$$

equipped with the reduced closed subscheme $Z' = \text{Spec}(B/(t_1 \cdots t_r))$ for some $2 \leq s \leq r \leq d-1$. It is a straightforward calculation to check

- $\text{sing}(\text{Spec}(B))$ is the union of the integral closed subsets

$$E_{ij} = \text{Spec}(B/(u, v, t_i, t_j)) \simeq \mathbf{A}_k^{d-3}$$

for $2 \leq i < j \leq s$ (the affine space has coordinates $\{t_m\}_{1 \leq m \leq d-1, m \neq i, j}$),

- such pairs $(\text{Spec}(B), Z')$ satisfy the axiomatic Setup 30.2.1 apart from the projectivity requirement on X .

The final miracle comes in Appendix F: for $E \subset \text{sing}(X)$ an irreducible component and the blow-up $\phi : \text{Bl}_E(X) \rightarrow X$, the pair $(\text{Bl}_E(X), \phi^{-1}(Z))$, satisfies the axioms of Setup 30.2.1 with one less irreducible component in $\text{sing}(X)$. More precisely, the irreducible components of the singular locus of $\text{Bl}_E(X)$ are the strict transforms of the irreducible components $E' \neq E$ of $\text{sing}(X)$ (for example, if $\text{sing}(X) = E$ topologically inside X then $\text{Bl}_E(X)$ is *smooth*). To verify this miracle, the only content is at k -points of $\text{Bl}_E(X)$ lying over E .

After finitely many such successive blow-ups we can pass to the case $X = X^{\text{sm}}$, so $Z = Z \cap X^{\text{sm}}$ is an ncd inside $X^{\text{sm}} = X$. Appendix E then provides a final modification $h : X' \rightarrow X$ centered over Z with smooth X' and $h^{-1}(Z) \subset X$ an sncd, so we are done!

Appendix A

Blow-up review

I first understood blow-ups from some notes of Deligne (at the start of his personal notes, in French, on Nagata compactifications that he shared). The following is an expanded version of that, partly borrowing from notes in an algebraic geometry course I once taught.

§A.1 Basic construction

Blowing up a variety along a closed subvariety, or a scheme along a closed subscheme, is one of the more important geometric operations in algebraic geometry. It arose historically in the study of “resolution of singularities”, but also occurs in quite different settings. There are at least 3 initial descriptions of blow-ups, most of which look uncomputable at first sight.

We will first look at what blow-ups do classically. Let X be a variety over $k = \bar{k}$. Then given the closed singular locus $Z \subset X$, we seek to “rip X apart along the tangent directions to Z ” to get the blow-up $\pi : \widetilde{X} \rightarrow X$ which is a birational morphism that is an isomorphism over $X - Z$: $\widetilde{X} - \pi^{-1}(Z) \simeq X - Z$. One image to keep in mind is that of a nodal curve, which we rip along the two tangent directions to get a smooth curve dominating it (with 2 points over the singularity).

There are three approaches:

- (1) (Classical definition): Take the schematic closure of $X - Z$ in an auxiliary variety. This method is typically only discussed in the classical setting of varieties (see [Har77, Ch. I]). It is a bit mysterious at first, and doesn’t initially look amenable to computations.
- (2) (Relative Proj construction): This is also unintuitive, and looks even more uncomputable, but it is technically useful. We will get to it in §A.2. This is the approach [Har77, Ch. II, Section 7] and in the abstruse [EGA, II, §8].
- (3) (Universal property): This approach is due to Hironaka (generalizing the classical blowing up at a point), and is rather computable if X is not too “bad” along Z . The blowup is the final object in a specific category of “nice” objects over X . We adopt this final approach as our starting point.

The setup is as follows: let $Z \hookrightarrow X$ be a closed immersion of schemes. Then consider those morphisms $f : Y \rightarrow X$ for which the base change closed immersion $f^{-1}Z \hookrightarrow Y$ is Cartier (where $f^{-1}Z := Y \times_X Z$). This says the quasi-coherent ideal sheaf $\mathcal{I}_Z \mathcal{O}_Y$ is invertible. (A boring example of this type is to take Y to be empty.)

We seek a *final object* $Y \xrightarrow{f} X$ among such X -schemes. It will be covered by affine open subschemes reminiscent of the open covering of $\mathbf{P}_X^n$ covered by the affine spaces $D_+(t_j)$ ’s which each refine the mapping property of $\mathbf{P}_X^n$. However, similarly to normalizations, the mapping property for

the construction we seek will only be as a final object among certain sufficiently nice X -schemes, not with a description of its functor of points on general X -schemes (so this is less robust than the mapping property for projective spaces).

DEFINITION A.1.1. The *blow-up* of X along Z is a morphism $\text{Bl}_Z(X) \xrightarrow{\pi} X$ which is final among maps $f : Y \rightarrow X$ with $f^{-1}Z \rightarrow \text{Bl}_Z X$ Cartier. The Cartier pullback $E = \pi^{-1}Z \hookrightarrow \text{Bl}_Z(X)$ is called the *exceptional divisor*.

How do we build this? Working locally, we focus on the affine case: $X = \text{Spec } A$, $Z = \text{Spec}(A/I)$. Consider any $Y \rightarrow X = \text{Spec}(A)$ such that the quasi-coherent ideal sheaf $\mathcal{I} = I\mathcal{O}_Y \subset \mathcal{O}_Y$ is invertible. Zariski-locally on Y , we can find an $\mathcal{O}_Y$ -basis of $\mathcal{I}$ given by elements $a \in I$ (i.e. on members of an open cover, we can trivialize $\mathcal{I}$ with a basis coming from I itself). Indeed, it is enough to check the y -stalks $\mathcal{I}_Y$ as an $\mathcal{O}_y$ -module for all $y \in Y$, since a basis of a stalk of a locally free sheaf of finite rank spreads out to a basis over an open neighborhood. Now use the following lemma:

LEMMA A.1.2. Let M be a free module of rank $d > 0$ over a local ring $(R, \mathfrak{m})$, and $\Sigma \subset M$ a (possibly infinite) generating set. Then there exists a basis of M in Σ .

Proof. Let $k = R/\mathfrak{m}$, so the map $\Sigma \rightarrow M/\mathfrak{m}M \simeq k^{\oplus d}$ has image that spans. Then by linear algebra, there is subset $\sigma_1, \dots, \sigma_d \in \Sigma$ such that $\{\sigma_i\} \subset M$ reduces to a k -basis. This is then an R -basis of the free module M by Nakayama's lemma. $\square$

In particular, Y has the open cover $\{Y_a\}_{a \in I}$ where

$$Y_a = \{y \in Y : a \text{ is an } \mathcal{O}_{Y,y}\text{-basis of } I\mathcal{O}_{Y,y}\}.$$

We can do better when $I = (a_1, \dots, a_m)$ is finitely generated (as is usually the case): then the finite collection of opens $\{Y_{a_1}, \dots, Y_{a_m}\}$ covers Y (this should be regarded as an analogue of $\{D_+(t_j)\}$ covering $\mathbf{P}_A^n$, for reasons which will be clearer towards the end of this section).

Let us now choose $a \in I$ and impose on $Y \xrightarrow{f} X$ the refined condition that:

$$I\mathcal{O}_Y \text{ is an } \mathcal{O}_Y\text{-free with basis } \{a\}. \quad (*a)$$

(this is analogous in the case of $\mathbf{P}^n$ to the universal property of a single $D_+(t_i)$: it represents the condition on surjections $\mathcal{O}^{n+1} \twoheadrightarrow \mathcal{L}$ that the i th standard basis section of $\mathcal{O}^{n+1}$ for $0 \leq i \leq n$ is carried to a global basis of $\mathcal{L}$). We seek to find the final such Y subject to this refined condition. It is crucial that we are not merely imposing principality with generator a but really *freeness* with generator a .

PROPOSITION A.1.3. Consider the A -subalgebra $A[La^{-1}] \subset A_a$ generated by elements a'/a for $a' \in I$. Then $\text{Spec}(A[La^{-1}]) \rightarrow \text{Spec } A$ is final for $(\ast a)$.

Proof. Setting $A' = A[La^{-1}] \subset A_a$, we see that in A' the element a is *not* a zero-divisor since A' is a subring of A_a in which a is a unit, and for any $a' \in I$ we can write $a' = (a'/a)a$ with $a'/a \in A'$, so IA' is A' -free on $\{a\}$. So $\text{Spec } A'$ at least belongs to the desired category of A -schemes. We seek to show it is the final object in this category.

That is, for any $f : Y \rightarrow \text{Spec } A$ for which $I\mathcal{O}_Y$ is free with basis a , we seek to uniquely fill in a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\exists! ?} & \text{Spec } A' \\ & \searrow f & \downarrow \\ & & \text{Spec } A, \end{array}$$

Thanks to the uniqueness aspect, for this purpose we can work *locally* on Y . Hence, it suffices to treat the affine case $Y = \text{Spec } R$ (any two affine opens in the original Y have overlap covered by affine opens!). Note that by the hypothesis $IR = Ra$ is *free* with basis a , the element $a \in R$ is not a zero-divisor (even though a may well be a zero-divisor in A) This will be crucial.

Since a is not a zero divisor in R , the natural map $R \rightarrow R_a$ is *injective*. Then by considering the composite map $A \rightarrow R \hookrightarrow R_a$, we get an associated unique map of A -algebra $h : A_a \rightarrow R_a$. Let's assemble this information in a commutative diagram:

$$\begin{array}{ccccc} A' = A[La^{-1}] & \xrightarrow{?} & R & \xrightarrow{!!} & R_a \\ & \nwarrow & \uparrow & \exists! h \uparrow & \\ & & A & \longrightarrow & A_a \end{array}$$

The key point is that the A -subalgebra $h(A') \subset R_a$ lies in R precisely because $IR = Ra$!

Thus, we get an A -algebra map $A' \xrightarrow{\varphi} R$, and to check uniqueness consider the commutative diagram of A -algebras

$$\begin{array}{ccc} A' & \xrightarrow{\varphi} & R \\ \downarrow & & \downarrow \\ A'_a = A_a & \longrightarrow & R_a \end{array}$$

where the bottom map is φ_a . But we also noted that $h : A_a \rightarrow R_a$ is the unique A -algebra map, so the bottom horizontal map φ_a is really h , no matter what φ may exist. The vertical maps are injections, so we have the desired uniqueness for φ . $\square$

DEFINITION A.1.4. For $a \in I$, we define $D_+(a) = \text{Spec}(A[La^{-1}])$.

This notation is meant to be suggestive of $D_+(f) \subset \text{Proj } S$, and below it will relate nicely to the standard affine charts of $\mathbf{P}_A^N$ when I is finitely generated. For $a, a' \in I$, we have the diagram of

A -schemes

$$D_+(a) \subset D_+(a)_{a'/a} \xrightarrow{\alpha} D_+(a')_{a/a'} \supset D_+(a').$$

where α over A between the open subschemes $D_+(a)_{a'/a}$ and $D_+(a')_{a/a'}$ arises from each being final for the same condition on an A -schemes Y : that $I\mathcal{O}_Y$ has each of a and a' as a *basis* (check this shared finality property!). This isomorphism α just says concretely that inside the localization $A_{aa'}$, we have an actual equality of A -subalgebras

$$A[La^{-1}]_{a'/a} = A[La'^{-1}]_{a/a'}.$$

Checking the triple overlap condition is easy since it expresses an equality among three A -subalgebras of $A_{aa'a''}$, so we can glue the $D_+(a)$'s along such α 's.

PROPOSITION A.1.5. This gluing is a final object for the Cartier condition on $I\mathcal{O}_Y$ which defines $\mathrm{Bl}_Z(X) = \mathrm{Bl}_I(A)$.

Proof. The argument is very similar to how the universal property of the gluing $\mathbf{P}_A^n$ is established using the meaning of each $D_+(t_j) \subset \mathbf{P}_A^n$ and their overlaps (check!). $\square$

§A.2 Computing blow-ups

If $I = (a_0, \dots, a_m)$ then $\mathrm{Bl}_I(A)$ is covered by the affine open subschemes $D_+(a_j) = \mathrm{Spec} A[La_j^{-1}]$. Soon we will see (Example A.2.2) that there is a closed immersion

$$i : \mathrm{Bl}_I(A) \hookrightarrow \mathbf{P}_A^m$$

over A such that $i^{-1}(D_+(t_j)) = D_+(a_j)$.

In general there are then two ways of trying to “compute” $D_+(a)$.

Method 1: If $I = (a_1, \dots, a_m)$, we aim to describe the $D_+(a_j)$ in terms of a ring

$$A_j = A[T_{ij}]_{i \neq j} / (a_i - T_{ij}a_j)_{i \neq j},$$

the initial A -algebra where each a_i for $i \neq j$ is a *specified* multiple of a_j . We want a *unique* multiplier. The issue is that the a_j may be a zero divisor in this quotient ring (even if A is a domain). To fix this, we kill all a_j -power torsion: $A_j / (a_j^\infty\text{-torsion})$. (Beware it is not enough to kill a_j -torsion, since there could be a_j^2 -torsion which gives rise to nonzero a_j -torsion in the quotient by the a_j -torsion. For instance, the quotient of the ring $R = \mathbf{Z}/(27)$ by its 3-torsion ideal $9R$ is $\mathbf{Z}/(9)$ in which there is nonzero 3-torsion.)

This quotient is precisely $A[La_j^{-1}]$, since they satisfy the same initial property among A -algebras (initial on the ring side expresses finality on the scheme side). The description as such a quotient of A_j is related to the classical description of the blowup, which we'll discuss tomorrow. How does

one compute this a_j^∞ -torsion concretely? It can be pretty difficult in general, and the difficulty of that computation is one way of expressing the difficulty of computing the schematic closure in the classical description of blow-ups (to be seen a bit later below).

Method 2: We claim that $\left(\bigoplus_{n \geq 0} I^n\right)_{a_1} = A[La^{-1}]$, where $a_1 := a \in I$ is viewed in degree 1 (hence the subscript 1).

Proof. We have $\left(\bigoplus_{n \geq 0} I^n\right)_{a_1} \subset \left(\bigoplus_{n \geq 0} A\right)_{a_1} = A[T]_{(aT)}$, and this is the A -subalgebra generated by degree-0 fractions b/a^r for $b \in I^r$. But any such b is a finite A -linear combination of products of r elements of I , so this A -subalgebra of A_a is generated by the fractions a'/a for $a' \in I$. This in turn is exactly how we defined $A[La^{-1}]$. $\square$

In the A -scheme $B = \text{Proj}\left(\bigoplus_{n \geq 0} I^n\right)$, the preceding description provides open subschemes $D_+(a) = \text{Spec}(A[La^{-1}])$ for $a \in I$ (viewed in degree 1 for the $\mathbf{N}$ -graded A -algebra $\bigoplus_{n \geq 0} I^n$), and these are certainly a cover (since $\bigoplus_{n \geq 0} I^n$ is generated by its degree-1 part I over its degree-0 part A). Moreover, their overlaps are the *same* as in the gluing $\text{Bl}_I A$. This shows:

THEOREM A.2.1. $\text{Bl}_I(A) = \text{Proj}(\bigoplus_{n \geq 0} I^n)$.

This Proj description is the *definition* of the blow-up in [Har77, Ch. II, §7] (and in other references). It is efficient, but quite unmotivated as an initial definition. The approach we have taken hopefully provides a more concrete understand of why this Proj construction has the desired finality property for the blow-up. Next time we will globalize beyond the case of affine X and discuss some properties and work out some calculations with the “affine chart” perspective developed above.

Here is a nice application of the Proj description:

EXAMPLE A.2.2. In the case $I = (a_1, \dots, a_m)$, we have a closed immersion $\text{Bl}_I A \hookrightarrow \mathbf{P}_A^{m-1}$ over A via the surjection of $\mathbf{N}$ -graded A -algebras $A[T_1, \dots, T_m] \rightarrow \bigoplus_{n \geq 0} I^n$ given by $T_j \mapsto (a_j)_1$.

By definition $\pi : \text{Bl}_Z(X) \rightarrow X$ is a relative Proj, namely $\text{Proj}(\mathcal{S})$ for the $\mathbf{N}$ -graded quasi-coherent $\mathcal{O}_X$ -algebra $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{S}^n$ for the quasi-coherent ideal $\mathcal{S} \subset \mathcal{O}_X$ of $Z \subset X$. The compatibility of relative Proj with any base change thereby gives an explicit description of the exceptional divisor $E = \pi^{-1}(Z)$:

$$E = \text{Proj}((\mathcal{O}_X/\mathcal{S}) \otimes_{\mathcal{O}_X} (\bigoplus_{n \geq 0} \mathcal{S}^n)) = \text{Proj}(\bigoplus_{n \geq 0} \mathcal{S}^n / \mathcal{S}^{n+1}).$$

In the special case that X is smooth over a field k and Z is also smooth over k , this takes on a more concrete form, as we now explain.

LEMMA A.2.3. If X is smooth over a field k and Z is smooth over k as well then the natural

map

$$S(\mathcal{I}/\mathcal{I}^2) \rightarrow \bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$$

between graded $\mathcal{O}_Z$ -algebras arising from the universal property of symmetric algebras and the identity map on $\mathcal{I}/\mathcal{I}^2$ in degree 1 is an isomorphism. Moreover, $\mathcal{I}/\mathcal{I}^2$ is a vector bundle over $\mathcal{O}_Z$.

Proof. We may and do extend scalars to arrange that k is algebraically closed (so all closed points are k -rational). The problem is Zariski-local on X , so we can assume $X = \operatorname{Spec}(A)$ is affine. Let $I \subset A$ be the ideal of Z , so by hypothesis both A and A/I are regular. Our aim is to show that the natural map

$$S(I/I^2) \rightarrow \bigoplus_{n \geq 0} I^n/I^{n+1}$$

of A/I -algebras is an isomorphism, which is to say that $f_n : S^n(I/I^2) \rightarrow I^n/I^{n+1}$ is an isomorphism for all $n \geq 2$ (the cases $n = 0, 1$ are trivial), and that I/I^2 is locally free as an A/I -module (or equivalently by the noetherian property: it has free stalks at maximal ideals). The map is easily seen to be surjective, but injectivity is rather less clear.

We are now going to carry out an algebraic analogue of the method of passing to “local coordinates” in differential geometry to reduce to a concrete situation where one can compute things. This rests on the technique of completion for noetherian rings along ideals, which is developed in [Mat, §8], especially Theorem 8.8 there.

It suffices to check after localizing at each maximal ideal $\mathfrak{m}$ of A/I . For both the freeness (equivalently flatness) of $(I/I^2)_{\mathfrak{m}}$ over $(A/I)_{\mathfrak{m}}$ and the isomorphism property of $(f_n)_{\mathfrak{m}}$ between finitely generated $(A/I)_{\mathfrak{m}}$ -modules, it suffices to check after faithfully flat base change to the completion of this local noetherian ring.

That completion is R/J where $R = A_{\mathfrak{m}}^{\wedge}$ is regular and $J = IR$ is an ideal of R for which R/J is also regular. The induced map of R -modules in degree n is $S^n(J/J^2) \rightarrow J^n/J^{n+1}$. Since R is a *complete* regular local noetherian k -algebra with residue field k , we have $R \simeq k[[t_1, \dots, t_d]]$ is a formal power series ring over k . Indeed, by regularity the maximal ideal has a set of d generators where $d = \dim R$, so that defines a local k -algebra map $k[t_1, \dots, t_d]_{(t_1, \dots, t_d)} \rightarrow R$, and by completeness of the target this factors through the completion as a local k -algebra map $k[[t_1, \dots, t_d]] \rightarrow R$. By completeness and separatedness of the target and completeness of the source, this is surjective due to a successive approximation argument (check, building up from the isomorphism property modulo squares of the maximal ideals). But both source and target are domains with the same dimension d , so the prime kernel ideal must be (0) .

The regular quotient R/J is also a formal power series ring, and we can arrange the choice of t_j ’s (which lifted any basis of $\mathfrak{m}/\mathfrak{m}^2$) so that the first m of them map to formal variables for a power series description of R/J . By surjectivity we can then easily change the other t_j ’s so they are all killed by the map, which is to say $J = (t_{m+1}, \dots, t_d)$. But now the map of interest $S^n(J/J^2) \rightarrow J^n/J^{n+1}$

over $R/J = k[[t_1, \dots, t_m]]$ is obtained by passing to completion at the origin on the analogous map for the usual polynomial ring $k[t_1, \dots, t_d]$ and its ideal $(t_{m+1}, \dots, t_d)$. In other words, by completion of a situation with abstract smooth k -schemes we have reduced our task to the special case of affine space and a linear subspace defined by the vanishing of some standard coordinates.

To summarize, we have reduced the general problem to the special case $X = \mathbf{A}_k^d$ (i.e., $A = k[t_1, \dots, t_d]$) and $Z = \{t_{m+1} = \dots = t_d = 0\} = \mathbf{A}_k^m$ (i.e., $I = (t_{m+1}, \dots, t_d)$). In this case by inspection I/I^2 as a module over $A/I \simeq k[t_1, \dots, t_m]$ is *free* of rank $d - m$, with basis $t_{m+1}, \dots, t_d$. Thus, $S^n(I/I^2)$ is also a free A/I -module, with basis given by monomials of degree n in $t_{m+1}, \dots, t_d$. But it is clear by inspection that I^n/I^{n+1} is not only spanned over A/I by those monomials but is even *freely* spanned, since the subring $k[t_1, \dots, t_m]$ of A maps isomorphically onto A/I . $\square$

By definition $\mathbf{P}(\mathcal{E}) := \text{Proj}(S(\mathcal{E}))$, so we immediately obtain:

PROPOSITION A.2.4. If X is smooth over a field k with dimension d and Z is also smooth over k with dimension $m < d$ then the exceptional divisor of $\text{Bl}_Z(X) \rightarrow X$ is a $\mathbf{P}^{d-m-1}$ -bundle over Z given by $\mathbf{P}(\mathcal{I}/\mathcal{I}^2)$, with $\mathcal{I}/\mathcal{I}^2$ a vector bundle over Z with rank $d - m$.

EXAMPLE A.2.5. Let's work out explicitly the most classical case of all: the blow-up of $\mathbf{A}_k^n$ at the origin. In effect, we will show that our definition of blow-up recovers the ad-hoc definition given near the end of [Har77, Ch. I, §4]. Since the ideal $I \subset k[t_1, \dots, t_n]$ for the origin is $(t_1, \dots, t_n)$, which has n generators, we have seen via Example A.2.2 that $\text{Bl}_0(\mathbf{A}_k^n)$ is a closed subscheme of $\mathbf{P}_{k[t_1, \dots, t_n]}^{n-1} = \mathbf{A}_k^n \times_k \mathbf{P}_k^{n-1}$. In [Har77, Ch. I, §4] the blow-up at the origin is *defined* by the system of homogenous linear equations over $k[t_1, \dots, t_n]$ given by $t_i x_j = t_j x_i$ for $i \neq j$ where $x_1, \dots, x_n$ are the homogeneous coordinates on $\mathbf{P}_k^{n-1}$. So we will show that closed subscheme is exactly $\text{Bl}_0(\mathbf{A}_k^n)$ as such a closed subscheme.

By design, inside the chart

$$D_+(x_i) = \mathbf{A}_{k[t_1, \dots, t_n]}^{n-1} = \text{Spec}(k[t_1, \dots, t_n][x_1/x_i, \dots, x_n/x_i])$$

of projective $(n - 1)$ -space, the closed subscheme for the blow-up is given by first imposing the relations $t_j = (x_j/x_i)t_i$ for $j \neq i$ (as x_j/x_i for $j \neq i$ are the standard coordinates on the affine $(n - 1)$ -space $D_+(x_i)$ over the base ring $k[t_1, \dots, t_n]$) and then killing all t_i^∞ -torsion. The relations $t_j = (x_j/x_i)t_i$ for $j \neq i$ are exactly the x_i -dehomogenization of the relations $t_i x_j = t_j x_i$, so it remains to check that upon imposing these dehomogenized relations the element t_i is not a zero-divisor (so there is no nonzero t_i^∞ -torsion to kill).

In other words, we want to show that t_i isn't a zero-divisor in the ring

$$k[t_1, \dots, t_n, x_1/x_i, \dots, x_n/x_i]/(t_j - (x_j/x_i)t_i)_{j \neq i}.$$

As a quotient of the polynomial ring, the imposed relations make the t_j 's disappear for $j \neq i$, leaving us with $k[t_i, x_1/x_i, \dots, x_n/x_i]$ that is visibly a domain in which t_i is nonzero, so indeed it isn't a zero-divisor in this ring.

§A.3 Strict transform

Let X be a scheme, $Z \subset X$ a closed subscheme with associated quasi-coherent ideal sheaf $\mathcal{I}$, $\pi : \widetilde{X} = \text{Bl}_Z(X) \rightarrow X$ the corresponding blow-up, and $Z' := \pi^{-1}(Z) = Z \times_X \widetilde{X} \subset \widetilde{X}$ the exceptional divisor (i.e., the closed subscheme corresponding to the invertible ideal $\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}$ on $\widetilde{X}$ via the universal property of blow-up). The restriction $\pi : \widetilde{X} - Z' \rightarrow U := X - Z$ is an isomorphism (since $\mathcal{I}|_U = \mathcal{O}_U$ is trivially invertible). In this way we view U as an open subscheme of $\widetilde{X}$ whose complement is defined by an invertible ideal. The inclusion $j : U \hookrightarrow \widetilde{X}$ is quasi-compact due to:

LEMMA A.3.1. Let $j : V \hookrightarrow Y$ be an open immersion of schemes whose complement is the zero locus of a quasi-coherent ideal of finite type (as is automatic when Y is locally noetherian). Then j is quasi-compact.

Proof. The task is local on Y , so we can assume $Y = \text{Spec}(A)$ is affine. Our task then reduces to showing that V is quasi-compact. The complement of V in Y is defined by a quasi-coherent ideal of finite type, and such an ideal has the form $\widetilde{I}$ for an ideal $I \subset A$ that is finitely generated. If $\{a_1, \dots, a_m\}$ is a finite generating set for I then the open complement V of $\text{Spec}(A/I)$ in $\text{Spec}(A) = Y$ is covered by the open subsets $\text{Spec}(A_{a_j})$ for $1 \leq j \leq m$. Hence, V is covered by finitely many affine open subschemes, so V is quasi-compact. $\square$

Since $j : U \hookrightarrow \widetilde{X}$ is quasi-compact, it has a meaningful scheme-theoretic image whose underlying space is the topological closure of U in $\widetilde{X}$ and whose corresponding quasi-coherent ideal sheaf is $\ker(\mathcal{O}_{\widetilde{X}} \rightarrow j_*(\mathcal{O}_U))$. But this kernel vanishes, since over open affines on $\widetilde{X}$ where the invertible ideal of the exceptional divisor is free of rank 1 the map j is $\text{Spec}(R_r) \rightarrow \text{Spec}(R)$ for $r \in R$ that isn't a zero-divisor, and then $\ker(R \rightarrow R_r) = 0$.

DEFINITION A.3.2. For a closed subscheme Y in X the *strict transform* $\widetilde{Y}$ of Y in X (relative to Z) is defined to be $\text{Bl}_{\mathcal{I} \cdot \mathcal{O}_Y}(Y) = \text{Bl}_{(\mathcal{I} + \mathcal{I}_Y)/\mathcal{I}_Y}(Y) = \text{Bl}_{Z \cap Y}(Y)$.

In the (uninteresting) case $|Y| \subset |Z|$ as underlying topological spaces inside $|X|$, the ideal $\mathcal{I} \cdot \mathcal{O}_Y$ consists of locally nilpotent sections, so the blow-up along this is empty (as the only way an ideal of nilpotent elements can be invertible is if one is in the zero ring). Our definition of “strict transform” is *not* the usual definition of such terminology; in Lemma A.3.3 we will prove the equivalence with another construction that is the usual definition.

The first key point is that the canonical map $\widetilde{Y} \rightarrow \widetilde{X}$ over $Y \hookrightarrow X$ (from functoriality of $\text{Bl}_Z(X)$ in the pair (X, Z) as described more broadly in the diagram (A.4.1) early in Section A.4)

is a closed immersion. Indeed, it suffices to check this over an open affine covering of X , and if $X = \operatorname{Spec}(A)$ is affine then $Z = \operatorname{Spec}(A/I)$ and $Y = \operatorname{Spec}(A/J)$ for ideals I and J in A , so $\tilde{Y} = \operatorname{Bl}_{I(A/J)}(\operatorname{Spec}(A/J)) = \operatorname{Bl}_{(I+J)/J}(\operatorname{Spec}(A/J)) = \operatorname{Bl}_{Y \cap Z}(Y)$. The natural map $\tilde{Y} \rightarrow \tilde{X}$ is (check!) induced by Proj “functoriality” applied to the map of graded A -algebras

$$\bigoplus_{n \geq 0} I^n \rightarrow \bigoplus_{n \geq 0} ((I+J)/J)^n = \bigoplus_{n \geq 0} (I^n + J)/J$$

that is surjective by inspection. This establishes that $\tilde{Y} \rightarrow \tilde{X}$ is a closed immersion.

We have shown that the strict transform of Y in X (relative to Z) is a closed subscheme of $\operatorname{Bl}_Z(X)$. Here is a description of this closed subscheme in a more direct manner (which is the usual definition of “strict transform”).

LEMMA A.3.3. Let X be a scheme, Y and Z closed subschemes of X , and $U = X - Z$ the open complement of Z in X . We view $Y \cap U$ as a closed subscheme of U , which in turn is an open subscheme of $\tilde{X} = \operatorname{Bl}_Z(X)$. The immersion $j : Y \cap U \rightarrow \tilde{X}$ is quasi-compact (even affine) and its scheme-theoretic closure in $\tilde{X}$ is the closed subscheme $\tilde{Y} := \operatorname{Bl}_{Z \cap Y}(Y)$.

In the special case that $Y \cap U$ is empty, which is to say $|Y| \subset |Z|$, this recovers the earlier observation that $\operatorname{Bl}_{Y \cap Z}(Y)$ is empty when $|Y| \subset |Z|$.

Proof. This can be checked locally on X , so we may assume $X = \operatorname{Spec}(A)$, $Z = \operatorname{Spec}(A/I)$, $Y = \operatorname{Spec}(A/J)$, so $\tilde{X} = \operatorname{Bl}_Z(X)$. The scheme $Y - (Z \cap Y) = Y \cap U$ is covered by open affines of the form $\operatorname{Spec}((A/J)_a)$, with $a \in I$. For $a \in I$, we let $Y_a = \operatorname{Spec}((A/J)_a)$ and $X_a = \operatorname{Spec}(A[Ja^{-1}])$. We have a canonical map

$$Y_a \rightarrow X_a \hookrightarrow \tilde{X}$$

(since $Y_a \subset U \subset \tilde{X}$ and the ideal I contains a which becomes a unit on Y_a , with X_a representing the condition that a freely generates $I\mathcal{O}_{\tilde{X}}$).

Let’s show $j^{-1}(X_a) = Y_a$ (so j is affine). We compute that for any $a' \in I$, $j^{-1}(X_a) \cap Y_{a'} \simeq Y_{a'} \times_{X_{a'}} (X_{a'} \cap X_a)$ is exactly

$$\operatorname{Spec}((A/J)_{a'} \otimes_{A[Ja'^{-1}]} A[Ja'^{-1}]_{a/a'}) = \operatorname{Spec}(((A/J)_{a'})_{a/a'}) = \operatorname{Spec}((A/J)_{aa'}).$$

This is exactly the open subscheme $Y_{a'} \cap Y_a$ inside of $Y \cap U$. Since $Y \cap U$ is covered by the open subschemes $Y_{a'}$ for $a' \in I$, taking the union over all $a' \in I$ yields $j^{-1}(X_a) = Y_a$, as desired.

The kernel of $\mathcal{O}_{\tilde{X}} \rightarrow j_*(\mathcal{O}_{Y \cap U})$ over X_a is exactly the quasi-coherent sheaf associated to the kernel of $A[Ja^{-1}] \rightarrow (A/J)_a = A_a/J_a$. Since this kernel is just $J_a \cap A[Ja^{-1}]$ (intersection inside A_a), we just need to show that it agrees with the kernel of $A[Ja^{-1}] \rightarrow (A/J)[((I+J)/J)a^{-1}]$. The target ring is a subring of $(A/J)_a = A_a/J_a$, so we’re done. $\square$

§A.4 Functoriality and base change for blow-ups

For a map of schemes $f : X' \rightarrow X$ and the closed subscheme $Z' := f^{-1}(Z) = Z \times_X X' \hookrightarrow X'$, consider the blow-ups

$$\pi' : \widetilde{X}' = \mathrm{Bl}_{Z'}(X') \rightarrow X', \quad \pi : \widetilde{X} = \mathrm{Bl}_Z(X) \rightarrow X.$$

By design, the closed subschemes $E' := \pi'^{-1}(Z') \subset \widetilde{X}'$ and $E := \pi^{-1}(Z) \subset \widetilde{X}$ are Cartier (the “exceptional divisor” for each blow-up). In particular, since $Z' = f^{-1}(Z)$, so $\pi'^{-1}(Z') = \pi'^{-1}(f^{-1}(Z)) = (f \circ \pi')^{-1}(Z)$, we see that under the composite map

$$f \circ \pi' : \widetilde{X}' \rightarrow X' \rightarrow X$$

the pullback of $Z \hookrightarrow X$ is Cartier. Hence, by the *universality property* (really the finality) of $\pi : \widetilde{X} \rightarrow X$ there is a unique map $\tilde{f} : \widetilde{X}' \rightarrow \widetilde{X}$ fitting into a commutative diagram

$$\begin{array}{ccc} \widetilde{X}' & \xrightarrow{\tilde{f}} & \widetilde{X} \\ \pi' \downarrow & & \downarrow \pi \\ X' & \xrightarrow{f} & X \end{array} \tag{A.4.1}$$

and by commutativity

$$\tilde{f}^{-1}(E) = \tilde{f}^{-1}(\pi^{-1}(Z)) = (\pi \circ \tilde{f})^{-1}(Z) = (f \circ \pi')^{-1}(Z) = \pi'^{-1}(f^{-1}(Z)) = \pi'^{-1}(Z') = E'.$$

In particular, the commutativity defines an X' -morphism

$$h : \widetilde{X}' \rightarrow X' \times_X \widetilde{X}$$

satisfying $h^{-1}(X' \times_X E) = h^{-1}(\mathrm{pr}_1^{-1}(E)) = (\mathrm{pr}_1 \circ h)^{-1}(E) = \tilde{f}^{-1}(E) = E'$.

It is natural to ask when h is an isomorphism, in which case we may say that the formation of the blow-up $\mathrm{Bl}_Z(X)$ *commutes with base change along f* . When f is an open immersion then h is an isomorphism; this expresses the evident fact from the construction that the formation of the blow-up is Zariski-local on X . A generalization of an open immersion is a flat map:

PROPOSITION A.4.1. If $f : X' \rightarrow X$ is flat then $h : \widetilde{X}' \rightarrow X' \times_X \widetilde{X}$ is an isomorphism. In general, for arbitrary f , h is a closed immersion.

Proof. The problem is Zariski-local on X and then on X' , so we may assume each is affine: $X = \mathrm{Spec} A$ and $X' = \mathrm{Spec} A'$ (so A' is an A -algebra), with Z corresponding to an ideal $I \subset A$ and Z' corresponding to the ideal $I' = IA' \subset A'$. the evident ring map $\varphi : S = \bigoplus_{n \geq 0} I^n \rightarrow \bigoplus_{n \geq 0} I'^n = S'$

over $A \rightarrow A'$ makes $\varphi(S_+)$ generate S'_+ over $S'_0 = A'$ since $I' = IA'$, so the “functoriality” of Proj yields an everywhere-defined morphism

$$\widetilde{X}' = \text{Proj}(S') \rightarrow \text{Proj}(S) = \widetilde{X}$$

over $\text{Spec}(A') \rightarrow \text{Spec}(A)$. By the uniqueness of $\widetilde{f}$, this map of Proj 's must be $\widetilde{f}$. Hence, since $X' \times_X \widetilde{X} = \text{Proj}(A' \otimes_A (\bigoplus_{n \geq 0} I^n)) = \text{Proj}(\bigoplus_{n \geq 0} (A' \otimes_A I^n))$, h is likewise the map of Proj 's induced by the map of graded A' -algebras

$$\bigoplus_{n \geq 0} (A' \otimes_A I^n) \rightarrow \bigoplus_{n \geq 0} I^n.$$

But this map of graded rings is surjective since $A' \otimes_A I \rightarrow IA' = I'$ is surjective, so the associated map h between their Proj 's is a closed immersion.

It remains to show that h is an isomorphism when f is flat, and for that purpose it suffices to show the map of graded A' -algebras giving rise to h is an isomorphism when A' is A -flat. Since $I^n = I^n A'$, we just need to show that for any ideal $J \subset A$, if A' is A -flat then the natural map $A' \otimes_A J \rightarrow JA'$ is an isomorphism. This latter map is certainly surjective, and for injectivity it suffices to analyze it after composing with the inclusion of JA' into A' . But that composite map is the result of applying the *exact* functor $A' \otimes_A (\cdot)$ to the A -linear injection of J into A , so we are done. $\square$

§A.5 Forcing disjointness via blow-ups

For a blow-up $q : \widetilde{X} = \text{Bl}_{\mathcal{I}}(X) \rightarrow X$ and a section s of $\mathcal{I}$ over an open subscheme $W \subset X$, we define $U(s)$ to be the open subscheme of $q^{-1}(W)$ over which $q^*(s)$, viewed as a section of $\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}$, is a local generator. (If $W = X = \text{Spec}(A)$ is affine, so $\mathcal{I} = \widetilde{I}$ for an ideal $I \subset A$, then $U(s)$ corresponds to what is called $D_+(s)$ in the construction of blow-ups in the affine case for an element $s \in I$.) We will sometimes abuse notation and speak of s , rather than $q^*(s)$, as a section of $\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}$ when there is no possibility of confusion.

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two quasi-coherent ideal sheaves on X and let $\widetilde{X}_i = \text{Bl}_{\mathcal{I}_i}(X)$. We now build a natural commutative diagram

$$\begin{array}{ccc} \widetilde{X}_{12} & \longrightarrow & \widetilde{X}_1 \\ \downarrow & & \downarrow \\ \widetilde{X}_2 & \longrightarrow & X \end{array} \tag{A.5.1}$$

in which $\widetilde{X}_{12} = \text{Bl}_{\mathcal{I}_1 \mathcal{I}_2}(X)$. First observe that $\widetilde{X}_{12}$ is covered by opens $U(s_1 s_2)$ for s_1 and s_2 respective sections of $\mathcal{I}_1$ and $\mathcal{I}_2$ over a common open W in X . On $U(s_1 s_2)$, neither s_1 nor s_2 is a zero divisor. For any section t_1 of $\mathcal{I}_1$ over W , $t_1 s_2$ is a section of $\mathcal{I}_1 \mathcal{I}_2 \cdot \mathcal{O}_{\widetilde{X}_{12}}$ over $U(s_1 s_2)$ and so $t_1 s_2$ is a multiple of $s_1 s_2$ here. Thus, t_1 is a multiple of s_1 , so $\mathcal{I}_1|_{U(s_1 s_2)}$ has s_1 as a basis. In this

way, we see that $\mathcal{I}_1 \cdot \mathcal{O}_{\widetilde{X}_{12}}$ is invertible on $\widetilde{X}_{12}$, and likewise with $\mathcal{I}_2$ replacing $\mathcal{I}_1$.

The existence of (A.5.1) then follows from the universal property of blow-ups, and the map $\widetilde{X}_{12} \rightarrow \widetilde{X}_j$ carries $U(s_1 s_2)$ into $U(s_j)$. Moreover, the resulting natural map from $\widetilde{X}_{12}$ to the blow-up of $\widetilde{X}_1$ along $\mathcal{I}_2 \cdot \mathcal{O}_{\widetilde{X}_1}$ carries $U(s_1 s_2)$ into $(U(s_1))(s_2)$, and for $W = \text{Spec}(A) \subset X$ the $U(s_1)$ -map $U(s_1 s_2) \rightarrow (U(s_1))(s_2)$ corresponds to the A -algebra map $(A[I_1^{-1} s_1])[I_2 s_2^{-1}] \rightarrow A[I_1 I_2 (s_1 s_2)^{-1}]$ that is an isomorphism by inspection (equality inside $A_{s_1 s_2}$). Since the $U(s_1 s_2)$'s (resp. the $(U(s_1))(s_2)$'s) cover $\widetilde{X}_{12}$ (resp. $\text{Bl}_{\mathcal{I}_2 \cdot \mathcal{O}_{\widetilde{X}_1}}(\widetilde{X}_1)$) as we vary W and s_1, s_2 , by using the known gluing data on such opens we thereby see that the natural map $\widetilde{X}_{12} \rightarrow \text{Bl}_{\mathcal{I}_2 \cdot \mathcal{O}_{\widetilde{X}_1}}(\widetilde{X}_1)$ is an isomorphism. Likewise, $\widetilde{X}_{12}$ is identified with the blow-up of $\widetilde{X}_2$ along $\mathcal{I}_1 \cdot \mathcal{O}_{\widetilde{X}_2}$.

In particular, by induction there are unique X -isomorphisms $\text{Bl}_{\mathcal{I}^n}(X) \simeq \text{Bl}_{\mathcal{I}}(X)$ for all $n \geq 1$; these isomorphisms are induced by applying relative Proj to the obvious explicit map on the level of quasi-coherent graded $\mathcal{O}_X$ -algebras.

PROPOSITION A.5.1. For closed subschemes Y and Z in a scheme X , the strict transform $\widetilde{Y}$ of Y in $\widetilde{X} = \text{Bl}_{Y \cap Z}(X)$ lies inside of the union of the $U(s)$'s as s ranges through all sections of $\mathcal{I}_Z$ over all opens.

Moreover, if $Y_1, \dots, Y_n$ are closed subschemes of X then the strict transforms $\widetilde{Y}_j$ in $\text{Bl}_{\cap Y_i}(X)$ satisfy $\cap \widetilde{Y}_j = \emptyset$.

Proof. We may assume $X = \text{Spec}(A)$ is affine. For the first claim, let $J = \Gamma(X, \mathcal{I}_Z)$ and $I = \Gamma(X, \mathcal{I}_Y)$. Clearly $\widetilde{X}$ is covered by $U(s)$ for $s \in I \cup J$ (as this union generates $I + J$), so it is enough to show that for $s \in I$, $U(s)$ is disjoint from the strict transform $\widetilde{Y} = \widetilde{V(I)}$ (as then $\widetilde{Y}$ is covered by the other $U(s)$'s, namely for $s \in J$). In terms of commutative algebra, the overlap of the A_s -scheme $U(s) = \text{Spec}(A[(I + J)s^{-1}])$ and the A -scheme $\widetilde{V(I)}$ in $\widetilde{X}$ is the affine scheme attached to the ring

$$A[(I + J)s^{-1}]/(I_s \cap A[(I + J)s^{-1}]),$$

with intersection $I_s \cap A[(I + J)s^{-1}]$ taken inside of A_s . But $1 \in I_s \cap A[(I + J)s^{-1}]$ since $s \in I$, so the overlap is empty.

For the second claim, let $I_j = \Gamma(X, \mathcal{I}_{Y_j})$, so since $\mathcal{I}_{\cap Y_j} = \sum \mathcal{I}_{Y_j}$ we have that the $U(s)$'s for $s \in \cup I_j$ cover $\widetilde{X} = \text{Bl}_{\cap Y_j}(X)$. It therefore suffices to show that for $s \in I_1$ the open set $U(s)$ is disjoint from $\widetilde{V(I_1)}$ (as then anything in every $\widetilde{V(I_j)}$ is outside of $U(s)$ for all $s \in \cup I_j$, yet such $U(s)$'s cover $\widetilde{X}$ and so there are no points in every $\widetilde{Y}_j$). This goes exactly as in the preceding calculation, setting $I = I_1$ and $J = I_2 + \dots + I_n$ (so $J = 0$ if $n = 1$). $\square$

Appendix B

Iterated Genericity

§B.1 Introduction

In the proof of Lemma 6.2.1, there is a genericity property whose proof deJong omits from his paper and which is essential for getting a smooth and geometrically connected fiber in the normal case. We first have to review the basic setup. Let X be d -dimensional closed subvariety in some projective space $\mathbf{P}^N$ with $1 \leq d < N$ (over an algebraically closed base field k), and let $Z \subset X$ be a reduced closed subscheme with pure dimension $d - 1$. We form a finite map $\pi : X \rightarrow \mathbf{P}^d$ by composition of $N - d$ linear projections away from points, and it is arranged to be generically étale with $\pi|_Z$ birational onto its image. Explicitly, $\pi = \pi_L$ is projection away from an $(N - d - 1)$ -plane $L \subset \mathbf{P}^N$ disjoint from X , where L is the projective span of independent k -points $\{p_1, \dots, p_{N-d}\}$ where p_1 is generic in $\mathbf{P}^N$, p_2 lifts a generic choice of k -point in the hyperplane target of projection away from p_1 , etc. (We will prove below that L 's arising in this way are generic in the Grassmann variety of $(N - d - 1)$ -planes in $\mathbf{P}^N$; this is *not* a tautology, even though each p_j is chosen generically within a projective space depending on the previous p_i 's!)

Geometrically, if L' is a fixed d -plane in $\mathbf{P}^N$ disjoint from the $(N - d - 1)$ -plane L then the projection $\mathbf{P}^N - L \rightarrow L'$ away from L sends a k -point z (such as $z \in X$) to the unique point where L' meets the $(N - d)$ -plane spanned by L and z . In coordinate-free and choice-free language (avoiding the non-canonical L'), if $\mathbf{P}^N = \mathbf{P}(V) := \text{Proj}(\text{Sym}(V))$ for an $(N + 1)$ -dimensional vector space V then $L = \mathbf{P}(V/W)$ for a $(d + 1)$ -dimensional subspace W and π is the restriction to X of the map $\mathbf{P}(V) - \mathbf{P}(V/W) \rightarrow \mathbf{P}(W)$ that carries the homothety class of a nonzero linear form $\ell \in V^\vee$ not vanishing in W (that is, not in $(V/W)^\vee$) to the homothety class of its nonzero restriction $\ell|_W$. This coordinate-free and choice-free description admits a functorial description over k -schemes (justifying algebraicity without needing to use coordinates), and this viewpoint will be required below when we work in a relative situation over some Grassmannians.

[We are using the contravariant Grothendieck convention for projective bundles and Grassmannians, so for example the projective space $\mathbf{P}(V)$ classifies families of hyperplanes in V , or equivalently isomorphism classes of families of 1-dimensional quotients of V . This is “dual” to the classical covariant viewpoint, but Grothendieck’s version is more convenient when working over base schemes as we shall have to do below. A very nice introduction to his perspective on these matters, as well as to the functorial picture and *construction* of Grassmannians and flag schemes in general, is Grothendieck’s elegant write-up in the complex-analytic category in [Gro60, Exp. 12]. The arguments there carry over to the case of schemes verbatim.]

Having made the finite (necessarily surjective) map $\pi : X \rightarrow \mathbf{P}^d$, we then chose a generic k -point $p \in \mathbf{P}^d - \pi(Z)$ such that π is étale over an open neighborhood of p and the finite projection

$\text{pr}_p : \pi(Z) \rightarrow \mathbf{P}^{d-1}$ to the “hyperplane” of lines through p is generically étale. In coordinate-free language, if $\mathbf{P}^N = \mathbf{P}(V)$ and $L = \mathbf{P}(V/W)$ then π has target $\mathbf{P}(W)$ and the point $p \in \mathbf{P}(W)$ is $\mathbf{P}(W/W_0)$ for a unique d -dimensional hyperplane $W_0 \subset W$. The projection pr_p has target given by the projective $(d-1)$ -space $\mathbf{P}(W_0)$. We introduced the incidence scheme

$$X' = \{(x, \ell) \in X \times \mathbf{P}(W_0) \mid x \in \pi^{-1}(\ell) = \ell \cap X\}$$

(points of $\mathbf{P}(W_0) = \mathbf{P}^{d-1}$ are identified with lines ℓ in $\mathbf{P}(W) = \mathbf{P}^d$ through p), and we considered the projection

$$f : X' \rightarrow \mathbf{P}(W_0) = \mathbf{P}^{d-1}.$$

For a point ℓ in the target (considered as a line through p in $\mathbf{P}(W) = \mathbf{P}^d$), the scheme-theoretic fiber $f^{-1}(\{\ell\})$ over the “point” $\ell \in \mathbf{P}(W_0)$ is the same as the scheme-theoretic preimage $\pi^{-1}(\ell)$ of the line $\ell \subset \mathbf{P}(W)$ under the finite surjection $\pi : X \rightarrow \mathbf{P}(W) = \mathbf{P}^d$. Let $H \subset \mathbf{P}(V) = \mathbf{P}^N$ be the $(N-d+1)$ -plane that is the closure of the subvariety of homothety classes of linear forms on V with a specified nonzero restriction to W_0 (corresponding to ℓ as a point of $\mathbf{P}(W_0)$), or more geometrically let H be the projective span the independent points $p_1, \dots, p_{N-d}$ and two points lifting a pair of independent points on the line $\ell \subset \mathbf{P}^d$. The scheme-theoretic fiber $f^{-1}(\{\ell\}) = \pi^{-1}(\ell)$ is precisely the linear slice $X \cap H$.

The question is this: as we vary through the permissible choices of $(N-d-1)$ -plane $L = \mathbf{P}(V/W)$, and then choices of the point $p = \mathbf{P}(W/W_0)$ in the projective space $\mathbf{P}(W)$ (which depends on $L = \mathbf{P}(V/W) \subset \mathbf{P}(V)$), are the resulting $(N-d+1)$ -planes H generic? That is, does this collection of H ’s at least exhaust some non-empty *open* set in the associated Grassmannian $\text{Gr}_d(V)$ of codimension- d subspaces in V (i.e., codimension- $(d+1)$ planes in $\mathbf{P}(V)$)? We need an affirmative answer because the Bertini theorem ensures that, with respect to the Zariski topology in $\text{Gr}_d(V)$, a generic $(N-d+1)$ -plane H has scheme-theoretic intersection $X \cap H$ that is a smooth and (geometrically) irreducible curve.

Intuitively, it must be the case that as we generically vary $L = \mathbf{P}(V/W)$ and *then* vary the point $p = \mathbf{P}(W/W_0) \in \mathbf{P}(W)$ subject to the various “generic” restrictions imposed upon it (namely, it lies outside of $\pi_L(Z) \subset \mathbf{P}(W)$, the projection $\text{pr}_p : \pi_L(Z) \rightarrow \mathbf{P}(W_0)$ is generically étale, and π_L is étale over an open neighborhood of p), the resulting $(N-d+1)$ -planes $H \in \mathbf{P}(V)$ “spanned” by L and a generic choice of line in $\mathbf{P}(W)$ through p should at least exhaust a non-empty open set in $\text{Gr}_d(V)$. If you reflect on the matter, you’ll see that there is something to be proved here because we have iterated genericity conditions: the conditions on the point p that lies in the projective space $\mathbf{P}(W)$ which *depends* on L , and the way that L was built up through iterated generic choices of k -points with each *depending on the preceding* generic choice. In the end we wish to sweep out at least a dense open locus in the Grassmannian $\text{Gr}_d(V)$ of H ’s in $\mathbf{P}(V)$.

It is probably the case that if you make artful non-canonical choices of planes in $\mathbf{P}^N$ to realize all projective spaces inside $\mathbf{P}^N$ then you can see the desired genericity by working entirely within

$\mathbf{P}^N$. However, at least for me the clearest way to do this rigorously is to avoid non-canonical choices and work directly with universal bundles over suitable Grassmannians. This is the approach that we shall take below.

§B.2 Genericity of L

As a first step toward genericity of H 's, we want to show that the L 's obtained in the initial part of the construction exhaust a Zariski-open locus in the variety $\mathrm{Gr}_{d+1}(V)$ of codimension $(d+1)$ -planes in the $(N+1)$ -dimensional vector space V (which is to say, codimension- d planes in $\mathbf{P}(V)$). That is, although we built our L in successive stages, what matters for our purposes is to have a dense open locus of such L 's. For any codimension- d plane $L = \mathbf{P}(V/W) \subset \mathbf{P}(V)$ disjoint from X such that the projection $\mathbf{P}(V) - L \rightarrow \mathbf{P}(W)$ has (necessarily finite and) generically étale restriction π_L to X , by choosing $N-d$ independent points $p_1, \dots, p_{N-d}$ in L we can realize π_L as a successive composite of projections away from the p_j 's. Hence, it is equivalent to prove:

THEOREM B.2.1. Let k be a field and V an $(N+1)$ -dimensional k -vector space. Let $X \subset \mathbf{P}(V)$ be a generically smooth closed subscheme with pure dimension $d < N$. Let $\mathbf{G} = \mathbf{Gr}_{d+1}(V)$ be the Grassmann variety of codimension- d planes in $\mathbf{P}(V)$. The locus of “points” $L = \mathbf{P}(V/W) \in \mathbf{G}$ such that L is disjoint from X and $\pi_L : X \rightarrow \mathbf{P}(W)$ is generically étale is dense and open.

In the final statement of the theorem it has to be understood that we are working with arbitrary field-valued points of the Grassmannian (not just k -points), and we are engaging in the usual abuse of notation by not explicitly writing in the base change to the field in which the point takes its values. (Of course, it is equivalent to work with points taking values in a fixed algebraically closed extension of k .)

Proof. We already know that this locus has a geometric point (and in fact has lots of geometric points). Since $\mathbf{G}$ is irreducible, the problem is therefore just to prove that this locus is open. By definition, $\mathbf{G}$ classifies isomorphism classes of quotient bundles of $V_S := V \otimes_k \mathcal{O}_S$ with rank $(N+1) - (d+1)$, which is to say subbundles of V_S with rank $d+1$ (for varying k -schemes S). In particular, there is a universal subbundle $\mathcal{E} \hookrightarrow V_{\mathbf{G}}$ of rank $d+1$ over $\mathbf{G}$, and so we get a “universal” $(N-d-1)$ -plane $\mathbf{P}(V_{\mathbf{G}}/\mathcal{E}) \hookrightarrow \mathbf{P}(V_{\mathbf{G}}) = \mathbf{P}(V) \times_k \mathbf{G}$ over $\mathbf{G}$. Consider the associated “universal projection” morphism

$$\pi : \mathbf{P}(V_{\mathbf{G}}) - \mathbf{P}(V_{\mathbf{G}}/\mathcal{E}) \rightarrow \mathbf{P}(\mathcal{E})$$

over $\mathbf{G}$ that is functorially defined as follows: to a rank- $(d+1)$ subbundle $\mathcal{F} \subset V_S$ and hyperplane bundle $\mathcal{H} \subset V_S$ fiberwise not containing $\mathcal{F}$ (so $\mathcal{F}$ corresponds to a k -map $\mathbf{Phi} : S \rightarrow \mathbf{G}$ and $\mathcal{H}$ corresponds to a $\mathbf{G}$ -map $S \rightarrow \mathbf{P}(V_{\mathbf{G}})$) we associate the hyperplane bundle $\mathcal{H} \cap \mathcal{F} \subseteq \mathcal{F}$. Over a geometric point $\mathrm{Spec} K \rightarrow \mathbf{G}$ corresponding to a $(d+1)$ -dimensional subspace $W \subset V_K = V \otimes_k K$,

the pullback of π is the usual projection $\mathbf{P}(V_K) - \mathbf{P}(V_K/W) \rightarrow \mathbf{P}(W)$ as considered earlier (given by restriction of homothety classes of linear functionals).

On geometric points, consider the condition on $\mathbf{G}$ that a codimension- d plane $\mathbf{P}(W)$ is disjoint from X . This is a *dense open* condition (stronger than merely containing a dense open set). Indeed, the overlap $X_{\mathbf{G}} \cap \mathbf{P}(V_{\mathbf{G}}/\mathcal{E})$ of closed subschemes in $\mathbf{P}(V_{\mathbf{G}}) = \mathbf{P}(V) \times_k \mathbf{G}$ has closed image in $\mathbf{G}$ by properness of projective bundles, and this closed set is not all of $\mathbf{G}$ because Bertini's theorem provides (many) codimension- $(d+1)$ planes in $\mathbf{P}(V)$ that do not meet the pure d -dimensional closed subscheme $X \subset \mathbf{P}(V)$. The complement of this closed set is a dense open $\mathbf{G}^0 \subset \mathbf{G}$, and it is precisely the locus of interest. All we have done is obtain an openness aspect in a Bertini theorem.

Now over $\mathbf{G}^0$ we have the inclusion $X_{\mathbf{G}^0} \subset \mathbf{P}(V_{\mathbf{G}^0}) - \mathbf{P}(V_{\mathbf{G}^0}/\mathcal{E}|_{\mathbf{G}^0})$, and so by restriction of π we arrive at the “universal” π_L :

$$\pi^{\text{univ}} : X_{\mathbf{G}^0} \rightarrow \mathbf{P}(\mathcal{E}|_{\mathbf{G}^0}) = \mathbf{P}(\mathcal{E})_{\mathbf{G}^0};$$

as we vary through geometric points of $\mathbf{G}^0$, this map gives precisely the projections π_L from X away from codimension- $(d+1)$ planes in $\mathbf{P}(V)$ that do not touch X . Since all maps π_L are finite, the map π^{univ} is finite on fibers over geometric points of $\mathbf{G}^0$, so π^{univ} is quasi-finite. It is also a $\mathbf{G}^0$ -map between proper $\mathbf{G}^0$ -schemes, so it is proper. Hence, π^{univ} is a finite map.

Letting X^{sm} be the dense open k -smooth locus of X , the quasi-finite restriction $\pi^0 : X_{\mathbf{G}^0}^{\text{sm}} \rightarrow \mathbf{P}(\mathcal{E})_{\mathbf{G}^0}$ of π^{univ} between flat $\mathbf{G}^0$ -schemes of finite type is fiberwise quasi-finite between smooth algebraic schemes of the same pure dimension. By the miracle flatness theorem (23.1 in Matsumura's CRT) such fibral restrictions are therefore flat, and so by the fibral flatness criterion it follows that the restriction of π^{univ} to $X_{\mathbf{G}^0}^{\text{sm}}$ is *flat*. Hence, the open étale locus U of π^0 meets each fiber over a geometric point $L \in \mathbf{G}^0$ in the étale locus of $\pi_L|_{X^{\text{sm}}}$. I claim that there is a dense open in the irreducible $\mathbf{G}^0$ over which U is fiberwise dense, and so this would provide a dense open locus of L 's for which π_L is generically étale on X .

It is harmless (check!) to make a finite extension on k , so we may assume that the connected components of X^{sm} are geometrically connected (and so geometrically irreducible). In this case, if $\{X_j^{\text{sm}}\}$ is the set of such components then the problem is to find a non-empty open $\mathbf{U}_j \subset \mathbf{G}^0$ over which the open set $U \cap (X_j^{\text{sm}})_{\mathbf{G}^0}$ meets all (irreducible!) fibers of $(X_j^{\text{sm}})_{\mathbf{G}^0}$. Indeed, such $\mathbf{U}_j$'s in the irreducible $\mathbf{G}^0$ must be dense and so they would have a common non-empty overlap that is the desired dense open locus of L 's in $\mathbf{G}^0$ over which U is fiberwise dense. To find the $\mathbf{U}_j$'s, first use the *definition* of $\mathbf{G}^0$ and deJong's 2.11 to find geometric points L of $\mathbf{G}^0$ for which $\pi_L|_{X_j^{\text{sm}}}$ is generically étale. Thus, $U \cap (X_j^{\text{sm}})_{\mathbf{G}^0}$ is a *non-empty* open set and hence its image under the *smooth* projection $(X_j^{\text{sm}})_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$ is a non-empty *open* set $\mathbf{U}_j$ that does the job. $\square$

To summarize, we have proved that in the Grassmannian $\mathbf{G}$, the condition on a geometric point $L = \mathbf{P}(V/W)$ that $L \cap X = \emptyset$ and $\pi_L : X \rightarrow \mathbf{P}(W)$ is generically étale is a *dense open* set. (This is stronger than just saying it contains a dense open set.) It remains to show that the further

requirement that π_L carries Z birationally onto its image in $\mathbf{P}(V)$ is a generic condition. Consider the universal case (using notation as in the previous proof)

$$\pi_Z^{\text{univ}} = \pi^{\text{univ}}|_{Z_{\mathbf{G}^0}} : Z_{\mathbf{G}^0} \rightarrow \mathbf{P}(\mathcal{E}|_{\mathbf{G}^0}).$$

This map is certainly finite (as it is proper and has finite restriction to geometric fibers over $\mathbf{G}^0$), and we seek a dense open locus in $\mathbf{G}^0$ over which it is birational onto its image on (geometric) fibers.

Consider the natural map of sheaves $\mathcal{O}_{\mathbf{P}(\mathcal{E}|_{\mathbf{G}^0})} \rightarrow \pi_{Z*}^{\text{univ}}(\mathcal{O}_{Z_{\mathbf{G}^0}})$. Since π_Z^{univ} is *finite* (hence affine), the formation of this map of sheaves commutes with any base change on $\mathbf{P}(\mathcal{E}|_{\mathbf{G}^0})$, so it commutes with any base change on $\mathbf{G}^0$. Thus, the formation of the *open* locus $U \subset \mathbf{P}(\mathcal{E}|_{\mathbf{G}^0})$ over which this sheaf map is surjective (use Nakayama for openness) is compatible with any base change on $\mathbf{G}^0$. Now by deJong's 2.11 and the definition of $\mathbf{G}^0$, there exists a geometric point $L \in \mathbf{G}^0(k)$ such that the open L -fiber $U_L \subset \mathbf{P}(\mathcal{E}|_L)$ contains a dense open in $\pi_L(Z)$, or equivalently (by finiteness of π_L and equidimensionality of Z) that the open $(\pi_Z^{\text{univ}})^{-1}(U) \subseteq Z_{\mathbf{G}^0}$ is dense in the L -fiber. We seek a non-empty open in the irreducible $\mathbf{G}^0$ over which $(\pi_Z^{\text{univ}})^{-1}(U)$ is dense in all fibers of $Z_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$.

Let $\{Z_j\}$ be the (reduced) irreducible components of the equidimensional Z over the algebraically closed field k , so by flatness of $(Z_j)_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$ the restriction $(\pi_Z^{\text{univ}})^{-1}(U) \cap (Z_j)_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$ to an open subset is also flat. This map is therefore open, but its image hits the geometric point L mentioned above and so its image is a *nonempty* open set. By varying j we get finitely many nonempty open sets in the irreducible $\mathbf{G}^0$, so they meet in a common dense open over which we have the desired fibral density in fibers of $Z_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$.

This completes the proof that for a generic $(N - d + 1)$ -plane $L = \mathbf{P}(V/W)$ in $\mathbf{P}(V) = \mathbf{P}^N$ we have $L \cap X = \emptyset$, $\pi_L : X \rightarrow \mathbf{P}(W) = \mathbf{P}^d$ is generically étale, and the restriction $\pi_L : Z \rightarrow \mathbf{P}(W) = \mathbf{P}^d$ is birational onto its image. In fact, the above proof gives a stronger property than genericity that will be essential in what follows: this collection of conditions on L is an *open* condition. The only aspect of such openness that has not been explained above is the generically étale property for π_L (given $L \cap X = \emptyset$; that is, $L \in \mathbf{G}^0$). But the finite π_L is generically flat for any L disjoint from X , so generic étaleness is equivalent to generic unramifiedness. This latter condition is precisely the generic vanishing of the coherent sheaf $\Omega_{X/\mathbf{P}(W)}^1$, and since X is (geometrically) irreducible it is equivalent that such vanishing occur *somewhere* on X . By Nakayama, it is equivalent to have a vanishing fiber at some point of X , and so by compatibility of formation of Ω^1 with respect to base change on the target we get the desired openness of this condition on L via Nakayama's Lemma. (Form the open set in $X_{\mathbf{G}^0}$ where its Ω^1 with respect to π^{univ} vanishes, and use the open image of this under the flat map $X_{\mathbf{G}^0} \rightarrow \mathbf{G}^0$.)

To summarize: not only have we proved the genericity of L in the Grassmannian $\mathbf{G} = \mathbf{Gr}_{d+1}(V)$, but we have shown that the conditions of interest on the codimension- d plane L in $\mathbf{P}(V)$ cut out exactly an open set $\mathbf{G}' \subset \mathbf{G}$.

§B.3 Genericity of H

Consider a codimension- d plane $L = \mathbf{P}(V/W) \subset \mathbf{P}(V)$ satisfying the preceding conditions. For each such L , consider points $p = \mathbf{P}(W/W_0) \in \mathbf{P}(W) - \pi_L(Z)$ such that (i) the finite generically étale map $\pi_L : X \rightarrow \mathbf{P}(W)$ is étale over an open neighborhood of p (that is, p lies outside of the closed set given by the π_L -image of the closed non-étale locus of π_L in the irreducible X), and (ii) the finite restriction $\mathrm{pr}_p : \pi_L(Z) \rightarrow \mathbf{P}(W_0)$ is generically étale. To p we associate the codimension- $(d-1)$ plane $H = \mathbf{P}(V/W_0)$ in $\mathbf{P}(V)$, and we want to prove that as we vary through all possible $L = \mathbf{P}(V/W)$ as above and the allowed $p \in \mathbf{P}(W)$, the resulting subspaces $H = \mathbf{P}(V/W_0)$ in $\mathbf{P}(V)$ sweep out at least a dense open in the corresponding Grassmannian G for V .

Consider the flag variety F classifying pairs $W_0 \subset W$ consisting of a $(d+1)$ -dimensional subspace $W \subset V$ and a hyperplane W_0 in W . This is a smooth projective variety, and it is equipped with two natural forgetful maps $F \rightarrow G$ and $F \rightarrow \mathbf{G}$ given by $(W_0, W) \mapsto W_0$ and $(W_0, W) \mapsto W$ respectively. Under the surjection $F \rightarrow G$ between irreducible varieties, the image of any nonempty open set contains a dense open. (In fact, this map is smooth, so the image of an open is open.) It is therefore sufficient to prove genericity in F for the set of pairs (W_0, W) as above over a nonempty open set in $\mathbf{G}$. We shall of course consider the part of F lying over the dense open $\mathbf{G}' \subset \mathbf{G}$ consisting of *precisely* the subspaces $L = \mathbf{P}(V/W)$ as above. Recall again the conditions on $p = \mathbf{P}(W/W_0)$: if $B_L \subset X$ denotes the closed nowhere dense non-étale locus (“branch scheme”) for the finite generically étale projection $\pi_L : X \rightarrow \mathbf{P}(W)$, then we want p to lie in $L - \pi_L(Z \cup B_L)$ and that $\mathrm{pr}_p : \pi_L(Z) \rightarrow \mathbf{P}(W_0) = \mathbf{P}^{d-1}$ is generically étale. Note that since the finite $\pi_L|_Z$ is generically étale (it is even birational, by the conditions on L), the hypothesis on pr_p is equivalent to requiring that $\mathrm{pr}_p \circ \pi_L : Z \rightarrow \mathbf{P}(W_0)$ is generically étale.

Aside from the condition that $p = \mathbf{P}(W/W_0)$ not lie in $\pi_L(B_L)$, the other requirements on $(W_0, W) \in F$ over the dense open $\mathbf{G}' \subset \mathbf{G}$ are that it lies in the part of F sitting over the dense open $G' \subset G$ defined with respect to Z exactly as the dense open $\mathbf{G}' \subset \mathbf{G}$ was defined with respect to X . (The openness of G' requires a moment’s reflection, since Z is reducible whereas X is irreducible. We were careful in Theorem B.2.1 to not impose irreducibility hypotheses so that it would carry over to Z in the present setting. For the other aspects it is straightforward to allow reducibility without losing openness or density.) It remains to check genericity in F for the locus cut out by the condition that $p = \mathbf{P}(W/W_0) \notin \pi_L(B_L)$ with $L = \mathbf{P}(V/W) \in \mathbf{G}'$. This locus is not empty because we have exhibited such L and p at the outset (the definitions of $\mathbf{G}'$ and G' did *not* involve ambiguous “genericity” restrictions!). It is therefore enough to prove abstractly that this locus is open in F .

Let $F^0 \subset F$ be the open preimage of $\mathbf{G}' \subset \mathbf{G}$, so the finite map

$$\pi_{F^0}^{\mathrm{univ}} : X_{F^0} \rightarrow \mathbf{P}(V_{F^0}/\mathcal{E}_{F^0})$$

over F^0 has open étale locus that is dense in all fibers over F^0 . The closed complement of this open

set has closed image in $\mathbf{P}(V_{F^0}/\mathcal{E}_{F^0})$, so the desired openness result follows from the rather general:

LEMMA B.3.1. Let $\mathcal{G}' \subset \mathcal{G}$ be a hyperplane bundle in a vector bundle $\mathcal{G}$ over a scheme S , and let $T \subset \mathbf{P}(\mathcal{G})$ be a closed subscheme. Let $j : S = \mathbf{P}(\mathcal{G}/\mathcal{G}') \hookrightarrow \mathbf{P}(\mathcal{G})$ be the canonical section. The condition on S -schemes that the pullback of S not meet the pullback of T is represented by an open subscheme of S .

Proof. The open locus $S - j^{-1}(T)$ does the job. □

Appendix C

Hilbert schemes and a “universal” stable marked curve

§C.1 Background on Hilbert schemes

Let $f : Y \rightarrow S$ be a proper finitely presented map of schemes. We seek to classify “flat families” of (finitely presented) closed subschemes. To be precise, consider the contravariant Set-valued functor $\mathrm{Hilb}_{Y/S}$ on S -schemes that assigns to each S -scheme T the set of finitely presented closed subschemes $Z \subset Y_T$ that are T -flat. (For a closed immersion $j : Z \hookrightarrow Y_T$, it is equivalent to say that j is finitely presented and that Z is finitely presented over T ; this avoids any possible confusion about what it means to say “ Z is finitely presented”. We leave the equivalence of conditions as an exercise; it uses only that Y_T is finitely presented over T .)

To be precise about regarding $\mathrm{Hilb}_{Y/S}(T)$ as a set, we observe that Z is determined up to *unique isomorphism* by the specification of its finitely presented quasi-coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_{Y_T}$, subject to the condition that $\mathcal{O}_{Y_T}/\mathcal{I}$ is T -flat. (Alternatively, a closed subscheme is determined uniquely up to unique isomorphism by its associated closed subset of Y_T and the stalks of its structure sheaf as quotients of those of $\mathcal{O}_{Y_T}$ at points of that closed subset.) The assignment $T \rightsquigarrow \mathrm{Hilb}_{Y/S}(T)$ is a functor by means of base change: for any $T' \rightarrow T$ over S , given any such $j : Z \hookrightarrow Y_T$ we get $j_{T'} : Z_{T'} \hookrightarrow Y_{T'}$ via base change.

One of Grothendieck’s first significant results on moduli schemes was to prove the representability of such Hilbert functors under projectivity conditions on Y over S (the name “Hilbert functor/scheme” due to the role of Hilbert polynomials in the result under projective hypotheses, as we will see below). A good reference for the material we review below, including a modern existence proof that incorporates Mumford’s simplifications in Grothendieck’s original proof, is [Nit05, Ch. 5]. (Grothendieck’s original proof in [his Bourbaki report](#) is very inspiring reading, but it is tough going in places since he omits some details that are not so easy for a beginner to fill in.)

What does it mean to say that $\mathrm{Hilb}_{Y/S}$ is representable? Or rather, for an S -scheme $\mathcal{H}$ what does it mean to specify an isomorphism of functors $\xi : \mathrm{Hom}_S(\cdot, \mathcal{H}) \simeq \mathrm{Hilb}_{Y/S}$? (That is, not just to say that $\mathcal{H}$ represents $\mathrm{Hilb}_{Y/S}$, but *how* it does so.) Evaluating ξ on $\mathcal{H}$ and chasing $\mathrm{id}_{\mathcal{H}}$ gives rise to an element of $\mathrm{Hilb}_{Y/S}(\mathcal{H})$, which is to say an $\mathcal{H}$ -flat finitely presented closed subscheme $\mathcal{Z} \subset Y \times_S \mathcal{H} =: Y_{\mathcal{H}}$, and by Yoneda-style general nonsense the condition that ξ is an isomorphism says precisely that this $\mathcal{Z}$ is *universal*: for any $T \rightarrow S$ and T -flat finitely presented closed subscheme

$j : Z \hookrightarrow Y_T$, there exists a *unique* S -map $q : T \rightarrow \mathcal{H}$ such that there is a Cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & \mathcal{Z} \\ j \downarrow & & \downarrow \\ Y_T & \xrightarrow{1_Y \times q} & Y_{\mathcal{H}} \end{array}$$

We stress that given q , α is *unique* if it exists at all, since α amounts to an isomorphism $Z \simeq Y_T \times_{1_Y \times q, Y_{\mathcal{H}}} \mathcal{Z} = T \times_{q, \mathcal{H}} \mathcal{Z}$ as closed subschemes of $Y_T = T \times_S Y = T \times_{q, \mathcal{H}} Y_{\mathcal{H}}$, any such isomorphism being unique if it exists since Z admits no nontrivial automorphism leaving j invariant. This is the reason that the specification of such q is all that matters (assuming some α exists at all), and is related to the fact that the objects being classified admit no nontrivial automorphisms (i.e., a T -automorphism of Z leaving j invariant must be id_Z); were that not the case then there is no reasonable way one could expect to have a universal structure $\mathcal{Z} \subset Y_{\mathcal{H}}$ classifying everything via solely the data of maps along the base (such as q above).

REMARK C.1.1. Grothendieck proved a remarkable functorial characterization [EGA, IV₃, 8.14.2] for when a map of schemes is locally of finite presentation (though no such criterion is known for being locally of finite type beyond the locally noetherian case, nor is there any known functorial criterion for being quasi-compact). This is not logically necessary here, but it explains why a representing object for the Hilbert functor must be locally of finite presentation over S if it exists at all.

As we will see, representing objects for Hilbert functors tend to be a countably infinite disjoint union of clopen subschemes, due to certain discrete invariants not controlled by the definition of the Hilbert functor (much as when considering line bundles on relative proper curves one needs to fix the fiberwise degree to cut out a bounded part of the relative Picard scheme).

To get a handle on the task of representing $\text{Hilb}_{Y/S}$, Grothendieck chopped this up into more manageable parts via the crutch of an auxiliary relatively ample line bundle. To be precise, we now put ourselves in the projective setting: suppose there exists a closed immersion $i : Y \hookrightarrow \mathbf{P}_S^n$ and define $\mathcal{L} = i^*(\mathcal{O}(1))$. This line bundle $\mathcal{L}$ is “ S -ample” (or “relatively ample”) on Y in any of the following three senses that are *equivalent* for any proper finitely presented map $Y \rightarrow S$:

- (i) there exists an open affine cover $\{U_\alpha = \text{Spec}(A_\alpha)\}$ of S such that $\mathcal{L}|_{Y_{U_\alpha}}$ is ample on the A_α -proper Y_{U_α} for all α ;
- (ii) for all open affine $U = \text{Spec } A \subset Y$, the restriction $\mathcal{L}|_{Y_U}$ is ample on the A -proper Y_U ;
- (iii) for all $s \in S$, the line bundle $\mathcal{L}_s$ on the $k(s)$ -proper Y_s is ample.

The equivalence of (i) and (ii) is elementary, but the equivalence with (iii) lies much deeper and is remarkable since there are no flatness hypotheses on Y over S (so one cannot access any base-change

theorems for coherent cohomology, say even if S were locally noetherian); see [EGA, IV₃, 9.6.4] for this amazing equivalence.

Let's now retain $\mathcal{L}$ and forget about i (which we won't need): we suppose there is given an S -ample $\mathcal{L}$ on Y in the above three equivalent senses (and we do not assume that $\mathcal{L}$ is globally generated by a finite set of global sections, as happens when $\mathcal{L}$ arises via pullback of $\mathcal{O}(1)$ as above). To decompose $\mathrm{Hilb}_{Y/S}$ into a disjoint union of “clopen subfunctors” via such an $\mathcal{L}$, we need two results on Euler characteristics. The first is [Mum12, Ch. II, §5, Cor. 1(b)] (proved there for a locally noetherian base; the general case is reduced to this by some limit techniques):

THEOREM C.1.2. Let $Z \rightarrow T$ be a proper finitely presented map, and $\mathcal{F}$ a finitely presented quasi-coherent sheaf on Z that is T -flat. Then the function on T defined by $t \mapsto \chi(\mathcal{F}_t) := \sum (-1)^i h^i(Z_t, \mathcal{F}_t)$ is locally constant for the Zariski topology. In particular, for varying $n \in \mathbf{Z}$ the pairwise disjoint subsets $\{t \in T \mid \chi(\mathcal{F}_t) = n\}$ of T are clopen.

This is a striking fact, since the individual terms $h^i(Z_t, \mathcal{F}_t) = \dim_{k(t)} H^i(Z_t, \mathcal{F}_t)$ are merely upper semi-continuous as t varies in T . The next result concerns the Euler characteristic of varying powers of a fixed line bundle:

THEOREM C.1.3. If Z is a proper scheme over a field and $\mathcal{L}$ is an invertible sheaf on Z then for any coherent sheaf $\mathcal{F}$ on Z the function $n \mapsto \chi(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is a polynomial in $\mathbf{Q}[n]$ with degree at most $\dim Z$.

Proof. In the projective case the result is addressed in many textbooks on algebraic geometry. The general case is deduced from the projective case via Chow's Lemma, as follows. By Grothendieck's Unscrewing Lemma [EGA, III₁, 3.1.2], to show that all coherent $\mathcal{F}$ satisfy the desired conclusion it suffices to show that for every irreducible closed subset $Z' \subset Z$ equipped with the reduced structure there exists some coherent $\mathcal{O}_{Z'}$ -module $\mathcal{G}$ with generic fiber of dimension 1 such that $\mathcal{G}$ viewed as an $\mathcal{O}_Z$ -module satisfies the desired conclusion. We will show that $\mathcal{G} = \mathcal{O}_{Z'}$ does the job.

Arguing by noetherian induction on the scheme Z , we may assume that for every proper closed subscheme $Z' \subset Z$ that is reduced and irreducible, every coherent $\mathcal{O}_{Z'}$ -module satisfies the desired conclusion when viewed as an $\mathcal{O}_Z$ -module. In particular, by the Unscrewing Lemma, we are done if Z is non-reduced or reducible (as the sufficient criterion above only involves closed subschemes of reduced irreducible components of Z_{red}), so we may assume Z is integral, and again via the Unscrewing Lemma we know that for every proper closed subscheme $Z' \subset Z$ (not necessarily irreducible!) and coherent $\mathcal{O}_{Z'}$ -module $\mathcal{G}$ the function $n \mapsto \chi(\mathcal{G} \otimes \mathcal{L}^{\otimes n})$ is in $\mathbf{Q}[n]$ with degree at most $\dim Z' \leq \dim Z$. By the Unscrewing Lemma one more time, it is now only necessary to exhibit a single coherent $\mathcal{O}_Z$ -module $\mathcal{F}$ with generic stalk of rank 1 such that $\chi(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is in $\mathbf{Q}[n]$ with degree at most $\dim Z$. We will show that $\mathcal{F} = \mathcal{O}_Z$ works.

By Chow's Lemma, there exists a proper birational map $\pi : Y \rightarrow Z$ with Y projective and integral. Thus, $\mathcal{O}_Z \rightarrow \pi_*(\mathcal{O}_Y)$ is an inclusion whose cokernel $\mathcal{Q}$ is supported on a proper closed

subscheme of Z (not necessarily irreducible!), so $n \mapsto \chi(\mathcal{Q} \otimes \mathcal{L}^{\otimes n})$ belongs in $\mathbf{Q}[n]$ with degree $< \dim Z$. Since

$$\chi(\mathcal{O}_Z \otimes \mathcal{L}^{\otimes n}) = \chi(\pi_*(\mathcal{O}_Y) \otimes \mathcal{L}^{\otimes n}) - \chi(\mathcal{Q} \otimes \mathcal{L}^{\otimes n}),$$

it suffices to show that $n \mapsto \chi(\pi_*(\mathcal{O}_Y) \otimes \mathcal{L}^{\otimes n})$ belongs to $\mathbf{Q}[n]$ with degree at most $\dim Z$.

Each higher direct image $R^j\pi_*(\mathcal{O}_Y)$ is coherent on Z with support on a proper closed subscheme for $j > 0$, so $n \mapsto \chi(R^j\pi_*(\mathcal{O}_Y) \otimes \mathcal{L}^{\otimes n})$ belongs to $\mathbf{Q}[n]$ with degree at most $\dim Z$ for $j > 0$. Since $R^j\pi_*(\mathcal{O}_Y) \otimes \mathcal{L}^{\otimes n} = R^j\pi_*(\pi^*\mathcal{L}^{\otimes n})$ for all $j \geq 0$, by the Leray spectral sequence

$$H^i(Z, R^j\pi_*(\cdot)) \Rightarrow H^{i+j}(Y, \cdot)$$

applied to $\pi^*\mathcal{L}^{\otimes n}$ we have

$$\chi(Y, \pi^*\mathcal{L}^{\otimes n}) = \sum_j (-1)^j \chi(Z, R^j\pi_*(\mathcal{O}_Y) \otimes \mathcal{L}^{\otimes n})$$

with the left side and all terms for $j > 0$ on the right side belonging to $\mathbf{Q}[n]$ with degree at most $\dim Z = \dim Y$. (We have used the settled projective case to handle the left side.) Thus, the term in this final sum for $j = 0$ satisfies the same property, and that is exactly what we needed. $\square$

Let's now return to our initial setup with the proper finitely presented $Y \rightarrow S$ equipped with an S -ample line bundle $\mathcal{L}$, and a T -flat finitely presented closed subscheme $Z \subset Y_T$ for an S -scheme T . Note in particular that $\mathcal{L}|_Z$ is T -flat. Thus, for $n \in \mathbf{Z}$ the function $t \mapsto \chi((\mathcal{L}|_Z)_t^{\otimes n})$ is locally constant on T . But this function of n is a polynomial of degree at most $\dim Z_t$, so it is determined by its values at the $1 + \dim Z_t$ integers $0 \leq n \leq \dim Z_t$. Since $t \mapsto \dim Z_t$ is locally constant on T (as $Z \rightarrow T$ is proper, flat, and finitely presented), for each $\Phi \in \mathbf{Q}[n]$ the pairwise disjoint sets

$$T_\Phi = \{t \in T \mid \chi((\mathcal{L}|_Z)_t^{\otimes n}) = \Phi(n) \text{ for all } n \in \mathbf{Z}\}$$

are all open (maybe likely empty!), so all are open and closed; note that T_Φ depends on the specification of $Z \subset Y_T$, despite the notation.

If we define $\text{Hilb}_{Y/S}^\Phi(T) = \{Z \in \text{Hilb}_{Y/S}(T) \mid \chi((\mathcal{L}|_Z)_t^{\otimes n}) = \Phi(n) \text{ for all } n \in \mathbf{Z}, t \in T\}$ then when $\text{Hilb}_{Y/S}$ is represented by some universal pair $(\mathcal{H}, \mathcal{Z})$ it follows easily (check!) that each clopen subscheme $\mathcal{H}_\Phi$ (defined via $\mathcal{Z} \subset Y_{\mathcal{H}}$) equipped with the restriction $\mathcal{Z}|_{\mathcal{H}_\Phi}$ represents $\text{Hilb}_{Y/S}^\Phi$. More importantly, the general clopen decomposition $\coprod_\Phi T_\Phi = T$ attached to any $Z \in \text{Hilb}_{Y/S}(T)$ implies (check!) that if there exist representing pairs $(\mathcal{H}_\Phi, \mathcal{Z}_\Phi)$ for $\text{Hilb}_{Y/S}^\Phi$ for all Φ (some such $\mathcal{H}_\Phi$ may be empty!) then

$$(\coprod_\Phi \mathcal{H}_\Phi, \coprod_\Phi \mathcal{Z}_\Phi)$$

represents $\text{Hilb}_{Y/S}$.

REMARK C.1.4. Even for the concrete case $\mathrm{Hilb}_{\mathbf{P}_{\mathbf{Z}}^N/\mathbf{Z}}$, it is a rather non-trivial matter to determine for which Φ the subfunctor $\mathrm{Hilb}_{\mathbf{P}_{\mathbf{Z}}^N/\mathbf{Z}}^\Phi$ is non-empty (equivalently, has a point valued in some algebraically closed field). Of course, a necessary condition is that Φ is a “numeric polynomial”, meaning that $\Phi(n) \in \mathbf{Z}$ for all $n \in \mathbf{Z}$; these are precisely the $\mathbf{Z}$ -linear combinations of the polynomials $n \mapsto \binom{n}{j}$ for $j \geq 0$.

The real theorem on this matter is:

THEOREM C.1.5 (Grothendieck). For any proper finitely presented $Y \rightarrow S$ and S -ample $\mathcal{L}$ on Y , each $\mathrm{Hilb}_{Y/S}^\Phi$ is represented by a proper and finitely presented S -scheme $\mathrm{Hilb}_{Y/S}^\Phi$ that is equipped with an S -ample line bundle. Hence, $\mathrm{Hilb}_{Y/S}$ is represented by the scheme $\coprod_\Phi \mathrm{Hilb}_{Y/S}^\Phi$.

The schemes $\mathrm{Hilb}_{Y/S}^\Phi$ are constructed as locally closed subschemes of Grassmannians Zariski-locally on S (then shown to be closed subschemes of such via the valuative criterion and not via the initial construction). A basic technical role for the T -flatness hypothesis on Z is that for all $T' \rightarrow T$ the natural surjection of $\mathcal{O}_{Y_{T'}}$ -modules

$$(\mathcal{I}_Z)_{T'} \rightarrow \mathcal{I}_{Z_{T'}}$$

is an isomorphism (since $0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_{Y_T} \rightarrow \mathcal{O}_Z \rightarrow 0$ retains short-exactness after any base change on T due to T -flatness of $\mathcal{O}_Z$).

As motivation for some considerations with marked stable curves, it is instructive to consider a refinement of the Hilbert functor:

DEFINITION C.1.6. Define $\mathrm{Hilb}_{Y/S}^{\mathrm{sec}}$ to be the contravariant functor that assigns to any S -scheme T the set of pairs (Z, σ) for $Z \in \mathrm{Hilb}_{Y/S}(T)$ and $\sigma \in Z(T)$.

There is an evident forgetful map $\mathrm{Hilb}_{Y/S}^{\mathrm{sec}} \rightarrow \mathrm{Hilb}_{Y/S}$ defined by $(Z, \sigma) \mapsto Z$. Thus, if $\mathrm{Hilb}_{Y/S}^{\mathrm{sec}}$ is to be representable, then a representing object naturally lives over the Hilbert scheme $\mathrm{Hilb}_{Y/S}$ (when the latter exists). In fact, it is a very specific object over the Hilbert scheme: the universal flat family! More precisely:

PROPOSITION C.1.7. Assume that $\mathrm{Hilb}_{Y/S}$ is represented by a pair $(\mathcal{H}, \mathcal{Z})$. Then $\mathrm{Hilb}_{Y/S}^{\mathrm{sec}}$ is represented by the S -scheme $\mathcal{Z}$.

Proof. Consider any S -scheme T and pair $(Z, \sigma) \in \mathrm{Hilb}_{Y/S}^{\mathrm{sec}}(T)$. The data of $Z \subset Y_T$ is classified by a unique S -map $f : T \rightarrow \mathcal{H}$, which is to say that $Z = \mathcal{Z} \times_{\mathcal{H}, f} T$ inside $Y_{\mathcal{H}} \times_{\mathcal{H}, f} T = Y_T$. Then the section $\sigma : T \rightarrow Z$ as a T -morphism amounts to a map $F : T \rightarrow \mathcal{Z}$ over the classifying S -map $f : T \rightarrow \mathcal{H}$ for Z . Clearly F is an S -map, and conversely given any S -map $F : T \rightarrow \mathcal{Z}$ we may define an S -map $f : T \rightarrow \mathcal{H}$ by composing F with the structure map $\mathcal{Z} \rightarrow \mathcal{H}$, with

such f defining an element $Z \in \mathrm{Hilb}_{Y/S}(T)$ via pullback of $\mathcal{Z}$ along f . In this way we see that specifying $(Z, \sigma) \in \mathrm{Hilb}_{Y/S}^{\mathrm{sec}}(T)$ is *exactly* the same as specifying an S -map $F : T \rightarrow \mathcal{Z}$, and this is all functorial in T . $\square$

It is a very instructive exercise to bootstrap the preceding argument to show that for any $n > 0$, the n -fold fiber power

$$\mathcal{Z} \times_{\mathcal{H}} \mathcal{Z} \times_{\mathcal{H}} \cdots \times_{\mathcal{H}} \mathcal{Z}$$

represents the functor $\mathrm{Hilb}_{Y/S}^{\mathrm{sec}, n}$ assigning to any S -scheme T the set of data $(Z; \sigma_1, \dots, \sigma_n)$ for $Z \in \mathrm{Hilb}_{Y/S}(T)$ and $\sigma_1, \dots, \sigma_n \in Z(T)$. (Note that the sections are given as an *ordered* n -tuple; this corresponds to making maps into the fiber product that has n ordered copies of $\mathcal{Z}$.) The open complement of the closed union of the pr_{ij} -pullback of the diagonal $\mathcal{Z} \subset \mathcal{Z} \times_{\mathcal{H}} \mathcal{Z}$ for all $1 \leq i < j \leq n$ represents the subfunctor of $\mathrm{Hilb}_{Y/S}^{\mathrm{sec}, n}$ corresponding to *pairwise disjoint* sections.

EXAMPLE C.1.8. Over a representing scheme $\mathcal{H}_n$ for $\mathrm{Hilb}_{Y/S}^{\mathrm{sec}, n}$, the condition that fibers be geometrically connected and semistable of pure dimension 1 with (arithmetic) genus g and the sections σ_i be pairwise disjoint is represented by an open subscheme $\mathcal{U}_n \subset \mathcal{H}_n$ over which we have a “universal genus- g curve” $(\mathcal{C}; s_1, \dots, s_n)$ inside Y with n disjoint markings. Imposing the further condition on the functor that each σ_i lands inside the relative smooth locus for the given flat family of curves is represented by the open subscheme of $\mathcal{U}_n$ complementary to the union of the closed subsets $s_i^{-1}(\mathcal{C} - \mathcal{C}^{\mathrm{sm}})$.

§C.2 Putting a stable marked curve into standard form

Let S be a scheme, and $f : X \rightarrow S$ a proper finitely presented flat map whose geometric fibers are semistable connected curves with arithmetic genus g . Let $\sigma_1, \dots, \sigma_n \in X^{\mathrm{sm}}(S)$ be pairwise disjoint sections supported in the smooth locus. Let $\mathcal{L}_{X/S} = \omega_{X/S}(\sum \sigma_i)$ be the canonically associated invertible sheaf with fiber-degree $2g - 2 + n$ which we assume to be *positive*. (Despite the notation, $\mathcal{L}_{X/S}$ depends on the σ_i ’s and not just on X over S .) By arguments in [DM69] and [Knu83], the geometric fibers are stable if and only if $\mathcal{L}_{X/S}$ is ample on all fibers X_s , and we now assume that to be the case.

By adapting cohomological vanishing arguments in the proof of [DM69, Thm. 1.2] (for $n = 0$ and $g \geq 2$), if $m \geq 2$ then $H^1(X_s, \mathcal{L}_{X_s/s}^{\otimes m})$ vanishes for $m \geq 2$ and $\mathcal{L}_{X/S}^{\otimes m}$ is fiberwise very ample for $m \geq 4$. (The reason we require $m \geq 4$ for fibral very ampleness whereas [DM69] only requires $m \geq 3$ is to handle certain degenerate cases with irreducible geometric fibers when $g \leq 1$.) The polynomial function

$$m \mapsto \chi(\mathcal{L}_{X_s/s}^{\otimes m})$$

has degree 1 with linear coefficient equal to the degree $2g - 2 + n$ and its value at $m = 0$ is

$\chi(\mathcal{O}_{X_s}) = 1 - g$, so

$$\chi(\mathcal{L}_{X_s/S}^{\otimes m}) = (2g - 2 + n)m + (1 - g).$$

Hence, if $m \geq 2$ then $h^0(X_s, \mathcal{L}_{X_s/S}^{\otimes m}) = (2g - 2 + n)m + (1 - g)$.

Defining $\Phi_{g,n}(t) = (2g - 2 + n)t + (1 - g) \in \mathbf{Q}[t]$, by Grothendieck's base change theorems for coherent cohomology we see that if $m \geq 2$ then $\mathcal{E}_m := f_*(\mathcal{L}_{X/S}^{\otimes m})$ is a vector bundle on S whose formation commutes with any base change and has fiber-rank $\Phi_{g,n}(m)$. Moreover, by the same base-change methods, the fibral very ampleness for $m \geq 4$ implies that the natural map $f^*(\mathcal{E}_m) \rightarrow \mathcal{L}_{X/S}^{\otimes m}$ is *surjective*. This quotient map is classified by an S -morphism

$$j_m : X \rightarrow \mathbf{P}(\mathcal{E}_m)$$

to a projective-space bundle, and the formation of j_m commutes with any base change on S . Thus, if $m \geq 4$ then j_m is a closed immersion on fibers over S , so the proper S -morphism j_m is quasi-finite and hence finite (by Zariski's Main Theorem). Thus, we can apply Nakayama's Lemma to deduce that j_m is a closed immersion.

Now fix $m = 4$ for specificity, and let $j = j_4$ and $\mathcal{E} = \mathcal{E}_4$ with rank

$$N := N(g, n) = 4(2g - 2 + n) + (1 - g);$$

this is a closed immersion of X into the $\mathbf{P}^{N-1}$ -bundle $\mathbf{P}(\mathcal{E})$ with fibers having Hilbert polynomial $\Phi(t) = \Phi_{g,n}(4t)$. In particular, if we Zariski-localize on S then we can also *choose* a trivialization

$$\mathcal{O}_S^{\oplus N} \simeq \mathcal{E}$$

and thereby identify j with a closed immersion $X \hookrightarrow \mathbf{P}^{N-1}$. Actually, we can get by with something *slightly less*: we can instead Zariski-localize so that there is merely an isomorphism $\varphi : \mathbf{P}(\mathcal{E}) \simeq \mathbf{P}^{N-1}$ of S -schemes (not necessarily arising from an isomorphism φ of vector bundles: $\mathrm{GL}_N(A) \rightarrow \mathrm{PGL}_N(A)$ is not surjective for non-local rings A in general!). Our aim in this handout is to reverse the entire process, by recovering a “universal” such stable n -pointed curve equipped with an appropriate projective embedding.

§C.3 Hilbert scheme considerations

In the above setup, (X, φ) gives us a map

$$h : S \rightarrow \mathrm{Hilb}_{\mathbf{P}^{N-1}/\mathbf{Z}}^{\Phi}$$

which classifies the closed immersion $j : X \hookrightarrow \mathbf{P}^{N-1}$. Let $U \subset \mathrm{Hilb}_{\mathbf{P}^{N-1}/\mathbf{Z}}^{\Phi}$ be the open subscheme over which the universal family has fibers that are connected semi-stable curves with arithmetic

genus g , and let $Z \rightarrow U$ be the universal such curve inside $\mathbf{P}_U^{N-1}$. The map h visibly factors through U , and the data of the ordered n -tuple of sections σ_i lifts $h : S \rightarrow U$ to a map $\tilde{h} : S \rightarrow Z^n$ (fiber power over U). The locus in Z^n where the n sections over U are pairwise disjoint and supported inside Z^{sm} is an open subset $\Omega \subset Z^n$, and $\tilde{h} : S \rightarrow Z^n$ lands inside Ω .

Over Ω there is a “universal semistable curve” C inside $\mathbf{P}_\Omega^{N-1}$, and by the miraculous [EGA, IV₃, 9.9.6] it is an *open* condition on Ω that $\mathcal{L}_{C/\Omega}$ has ample fibers. Thus, there is an open subscheme $H \subset \Omega$ over which there is a *universal* stable n -pointed genus- g curve $(C; \xi_1, \dots, \xi_n)$ equipped with the data of a specified closed immersion $\iota : C \hookrightarrow \mathbf{P}_H^{N-1}$. That is, $(C; \xi_1, \dots, \xi_n; i)$ represents the functor of stable n -pointed genus- g curves equipped with a closed immersion of the curve into $\mathbf{P}^{N-1}$. Letting $f : C \rightarrow H$ be the universal structure morphism, the vector bundle $\mathcal{F} := f_*(\mathcal{L}_{C/H}^{\otimes 4})$ of rank N has *nothing* to do with ι , and in particular $\mathcal{F}$ might not admit a trivialization over H .

Consider the Isom-scheme $I = \text{Isom}(\mathbf{P}(\mathcal{F}), \mathbf{P}^{N-1})$ over H . This is a Zariski-torsor for the automorphism scheme PGL_N of $\mathbf{P}^{N-1}$ (as we see by working Zariski-locally over H where $\mathcal{F}$ admits a trivialization), and by the very definition of I the pullback $\mathbf{P}(\mathcal{F})_I$ over I is equipped with a tautological isomorphism φ to $\mathbf{P}^{N-1}$. Hence, by composing the I -pullback of $j_4 : C \hookrightarrow \mathbf{P}(\mathcal{F})$ with φ , C_I is equipped with a canonical closed immersion $\iota' : C_I \hookrightarrow \mathbf{P}_I^{N-1}$. But this closed immersion has nothing to do with ι_I !

Finally, consider the condition on an I -scheme T that on the pullback C_T , the two closed immersions $C_T \rightarrow \mathbf{P}_T^{N-1}$ arising from ι_I and ι' coincide. If this condition is represented by a closed subscheme $B \subset I$ (i.e., $T \rightarrow I$ factors through B if and only if the pullbacks of ι_I and ι' over T coincide) then B represents the functor assigning to any scheme S the set of isomorphism classes of stable n -pointed genus- g curves $(X; \sigma_1, \dots, \sigma_n)$ equipped with an isomorphism $\varphi : \mathbf{P}(f_*(\mathcal{L}_{X/S}^{\otimes 4})) \simeq \mathbf{P}_S^{N-1}$ where $f : X \rightarrow S$ is the structure morphism and $N = N(g, n)$ as above. Here, by “isomorphism” between such data

$$(X; \sigma_1, \dots, \sigma_n, \varphi), \quad (X'; \sigma'_1, \dots, \sigma'_n, \varphi')$$

we mean an isomorphism

$$\theta : (X; \sigma_1, \dots, \sigma_n) \simeq (X'; \sigma'_1, \dots, \sigma'_n)$$

such that the *induced isomorphism*

$$f_*(\mathcal{L}_{X/S}^{\otimes 4}) \simeq f'_*(\mathcal{L}_{X'/S}^{\otimes 4})$$

has projectivization which carries φ to φ' . Note the crucial fact such combined data have *no non-trivial automorphisms* (so the task of determining an isomorphism class is actually *Zariski-local* on the base). To see this latter rigidity property, consider an automorphism α of $(X; \sigma_1, \dots, \sigma_n)$ whose effect on $\mathbf{P} := \mathbf{P}(f_*(\mathcal{L}_{X/S}^{\otimes 4}))$ intertwines the isomorphism φ with itself. The effect of α on $\mathbf{P}$ must then be the identity map, yet its effect on $\mathbf{P}$ recovers α on $X \subset \mathbf{P}$ due to the *canonicity* of the inclusion of X into $\mathbf{P}$ (depending on the ordered n -tuples of σ_i ’s that is respected by α).

We have now nearly proved the following theorem:

THEOREM C.3.1. Fix $n, g \geq 0$ such that $2g - 2 + n > 0$. Let $N = N(g, n)$ and $\Phi(t) = \Phi_{g,n}(4t)$ as defined above. The functor of isomorphism classes of stable n -pointed genus- g curves $(X; \sigma_1, \dots, \sigma_n)$ equipped with an isomorphism (or “trivialization”)

$$\varphi : \mathbf{P}(f_*(\mathcal{L}_{X/S}^{\otimes 4})) \simeq \mathbf{P}_S^{N-1}$$

is represented by a locally closed subscheme $\mathcal{M}_{g,n}^{\text{triv}}$ inside a PGL_N -bundle over an open subscheme of the n -fold fiber power of the universal flat family over the Hilbert scheme $\text{Hilb}_{\mathbf{P}_{\mathbf{Z}}^{N-1}/\mathbf{Z}}$.

It can be shown that the natural map from $\mathcal{M}_{g,n}^{\text{triv}}$ to the n -fold fiber power of the universal flat family over the Hilbert scheme is a locally closed immersion. We will not need this.

Proof. It remains to prove a general fact that has nothing to do with stable curves: if X, Y are closed subschemes of a projective space over a scheme T (in the case of interest Y is itself a projective space over T) with X flat over T , and if $f, h : X \rightrightarrows Y$ are two T -morphisms, then the condition on a T -scheme T' that $f_{T'} = h_{T'}$ is represented by a finitely presented closed subscheme of T .

One of the standard applications of the existence of Hilbert schemes is the existence of Hom-schemes by viewing a morphism $U \rightarrow V$ between finitely presented projective schemes over a base B as a graph inside $U \times_B V$, which is to say a finitely presented closed subscheme of $U \times_B V$ on which pr_1 is an isomorphism. Such a graph is flat over B when U is flat over B , so the place to look for a Hom-scheme is inside $\text{Hilb}_{(U \times_B V)/B}$. More specifically, as is explained in [Nit05, Ch. 5], the functor $T' \mapsto \text{Hom}_{T'}(X_{T'}, Y_{T'})$ is represented by a countable disjoint union H of finitely presented quasi-projective T -schemes (over which there is a “universal morphism” $X_H \rightarrow Y_H$). In particular, H is T -separated and the maps f and h correspond to a pair of T -morphisms $T \rightrightarrows H$, which is to say a section $j : T \rightarrow H \times_T H$ (which is a closed immersion since H is T -separated). The condition of equality $f_{T'} = h_{T'}$ after base change along a map $k : T' \rightarrow T$ is precisely the condition that $j \circ k : T' \rightarrow H \times_T H$ factors through the diagonal, which is to say that k factors through the closed subscheme $j^{-1}(\Delta_{H/T})$. Hence, this latter closed subscheme of T does the job. $\square$

The “difference” between the moduli scheme built in the above theorem and the dream of a universal stable n -pointed genus- g curve is the specification of φ . To remove φ from the picture, first note that there is a natural action of PGL_N on $\mathcal{M}_{g,n}^{\text{triv}}$ by composing φ with the PGL_N -action on $\mathbf{P}^{N-1}$. But this action is not a *free* action because two φ ’s for the same stable n -pointed genus- g curve can be related through the action of a non-trivial automorphism of the stable pointed curve. In other words, it is exactly the presence of nontrivial (though finite!) automorphism groups for some stable curves which prevents the action from being free.

The work of Artin [Art74] on moduli problems provides an appropriate notion of “quotient stack” $\mathcal{M}_{g,n}^{\text{triv}}/\text{PGL}_N$ that is a-priori an Artin stack. Due to the properties of automorphism schemes of

stable n -pointed genus- g curves over algebraically closed fields as in Lemma 14.1.1, it follows from [LMB, Thm. 8.1] that this stack is even Deligne-Mumford (informally, that means that the stack admits an étale cover by a scheme, a property that Deligne and Mumford proved directly in [DM69] via “slicing arguments” since at that time the notion of Artin stack didn’t exist and the notion of DM stack was only then being introduced).

In §17 and §18.1 this Deligne–Mumford stack is shown to be smooth and proper over $\mathbf{Z}$. Were the automorphism groups at geometric points actually trivial then it could be further concluded that this DM stack is an algebraic space, and there are useful sufficient criteria to prove that an algebraic space is a scheme. However, it is not an algebraic space due to the presence of non-trivial (though étale) automorphism schemes at geometric points, and in particular it is not a scheme.

Appendix D

Isomorphisms of quadratic spaces

§D.1 Introduction

Let A be a local ring, with residue field k , and $Q, Q' : A^n \rightrightarrows A$ two residually non-degenerate quadratic forms in n variables over A such that their reductions $q, q' : k^n \rightrightarrows k$ are isometric. (That is, there exists a linear automorphism L_0 of k^n such that $q \circ L_0 = q'$.)

Subject to the mild hypothesis $\text{char}(k) \neq 2$ when n is odd (i.e., we assume n is even if $\text{char}(k) = 2$), we aim to prove that if A is henselian (e.g., complete local noetherian, or the henselization of any local ring) then Q and Q' are isometric; i.e., we seek to build a linear automorphism L of A^n such that $Q \circ L = Q'$.

REMARK D.1.1. Let's show that the parity condition on n when $\text{char}(k) = 2$ cannot be removed. (We will only care about the case of even n , in fact $n = 2$, so the reader who doesn't care about residue characteristic 2 is welcome to skip this.) Suppose $\text{char}(k) = 2$ and $n = 2m + 1$ with $m \geq 0$.

For non-degenerate quadratic space (V, q) of odd dimension $2m + 1$ over such k , the symmetric bilinear form $B_q(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2)$ is alternating and has *defect space* $V^\perp := \{v \in V \mid B_q(v, \cdot) = 0\}$ equal to a line, with the induced alternating form $\bar{B}_q$ on the $2m$ -dimensional $V/V^\perp$ non-degenerate (i.e., symplectic.) Let

$$Q = x_0^2 + x_1x_2 + \cdots + x_{2m-1}x_{2m}, \quad Q' = ux_0^2 + x_1x_2 + \cdots + x_{2m-1}x_{2m}$$

for $u \in A^\times$. The reductions (k^n, q) and (k^n, q') have 1-dimensional defect spaces each coinciding with the line ke_0 .

The restrictions of q and q' to their respective defect lines are x_0^2 and $\bar{u}x_0^2$ respectively, so the existence of a residual isometry forces the reduction $\bar{u} \in k^\times$ to be a square in k . Since $\text{char}(k) = 2$, the reduction of u being a square does not generally imply that u is a square in $A^\times$ (even if A is a complete discrete valuation ring). Although it is clear by inspection that u being a square in $A^\times$ is sufficient for Q and Q' to be isometric, it isn't evident in general if this is *necessary* for Q and Q' to be isometric (in which case we would have a genuine obstruction, showing that the parity condition on n for residue characteristic 2 cannot be avoided).

In case A is an $\mathbf{F}_2$ -algebra then necessity is obvious because in such cases the intrinsic *defect modules* for Q and Q' respectively each coincide with the subbundle Ae_0 on which they respectively restrict to x_0^2 and ux_0^2 , so we can argue as we did over k . Thus, by taking A to be a local henselian $\mathbf{F}_2$ -algebra and $u \in A^\times$ a non-square unit whose reduction is 1 (e.g., $A = \kappa[[t]]$ for a field κ of characteristic 2 and $u = 1 + t$) we get the desired examples of non-isometric Q

and Q' that are residually non-degenerate and residually isometric for all odd $n = 2m + 1$.

In fact, by using more serious input through the structure of “odd” orthogonal group schemes over rings one can show over any local ring A that the condition of u being a square is always necessary for Q and Q' to be isometric; see Remark D.2.1. Thus, we also get mixed-characteristic examples with A any 2-adic integer ring, by taking u to be $1 + \pi$ for a uniformizer π (visibly a non-square precisely because if u were a square then it would have to have a square root that is a 1-unit but the square of any 1-unit is $1 \bmod \pi^2$ since $\pi|2$ in A).

§D.2 Smoothness of an Isom-scheme

Now we take up the task of proving for henselian local A that Q and Q' are isometric when they are residually isometric, provided that n is even when $\text{char}(k) = 2$. First we dispose of a boring case: $n = 1$. Suppose $n = 1$, so $Q = ax_0^2$ and $Q' = a'x_0^2$ for units $a, a' \in A^\times$ such that a'/a has reduction in $k^\times$ that is a square (by the residual isometry hypothesis). For odd n we are assuming $\text{char}(k) \neq 2$, so a'/a is therefore a square in A by the henselian condition (i.e., $Y^2 - a'/a \in A[Y]$ has a simple residual root, and thus a root in A) and so Q and Q' are isometric. Thus, we now assume $n \geq 2$.

The key idea is to introduce an A -scheme classifying isometries and prove this scheme is *smooth* (which will crucially use the hypothesis that n is even when $\text{char}(k) = 2$); it will then follow via the henselian condition on A that the given residual isometry $q' \simeq q$ can be lifted to an isometry $Q' \simeq Q$ over A .

Inside the A -group scheme GL_n , the condition on a point L that $Q' = Q \circ L$ is an explicit (albeit nasty) finite system of universal polynomial conditions over A on the matrix entries of L (depending on Q and Q' over A). This defines a finitely presented closed subscheme $I \subset \text{GL}_n$ representing the functor on A -algebras

$$C \rightsquigarrow \text{Isom}((C^n, Q'_C), (C^n, Q_C)).$$

We are given that $I(k)$ is non-empty. Thus, if I is A -smooth then $I(A) \rightarrow I(k)$ is surjective for henselian A , so we would be done. Our aim then is to prove that I is A -smooth. The beauty of this idea is that as a property of an explicit finitely presented A -scheme it will be sufficient to check this after suitable fpqc scalar extensions on A that would otherwise seem to lose all contact with our actual problem of interest over A .

It is discussed in Section 22.1 (with $n \geq 2$) that the residual non-degeneracy of Q implies that the projective quadric $(Q = 0) \subset \mathbf{P}_A^{n-1}$ is A -smooth with relative dimension $n - 2 \geq 0$, so we now forget about the assumptions that A is local and henselian (so in particular we drop the hypothesis involving a residual isometry!) and instead allow A to be any ring whatsoever but *assume* two things:

- (i) the projective quadrics $(Q = 0), (Q' = 0) \subset \mathbf{P}_A^{n-1}$ are A -smooth with relative dimension $n - 2$,

(ii) if n is odd then A is a $\mathbf{Z}[1/2]$ -algebra.

Under these assumptions, we shall prove that the Isom-scheme I is A -smooth. It is only at the end of the argument that (ii) will be used.

By [Con14, Lemma C.2.1] (whose proof is primarily a reference to a self-contained concrete calculation in [SGA7, XII, Prop. 1.2]), the smoothness of relative dimension $n - 2$ for the projective quadrics $(Q = 0)$ and $(Q' = 0)$ in $\mathbf{P}_A^{n-1}$ ensures that fppf-locally on $\mathrm{Spec}(A)$, each of Q and Q' becomes isometric to the “standard” fiberwise non-degenerate quadratic form in n variables, namely $Q_{2m} := x_1x_2 + \cdots + x_{2m-1}x_{2m}$ when $n = 2m$ and $Q_{2m+1} := x_0^2 + Q_{2m}$ when $n = 2m + 1$. Since the A -smoothness of the Isom-scheme I is an fppf-local problem over $\mathrm{Spec}(A)$, by making a suitable fppf-affine base change on A (!) we can assume $Q = Q' = Q_n$! Thus, our Isom-scheme I becomes the A -group scheme $O_{n,A} \subset \mathrm{GL}_{n,A}$ representing the functor of isometric automorphisms of Q_n . This A -group scheme is the scalar extension of the corresponding one over $\mathbf{Z}$, so our task has reduced to that of proving the smoothness of O_n over $\mathbf{Z}$ for even n and over $\mathbf{Z}[1/2]$ for odd n . (Recall that we assume A is a $\mathbf{Z}[1/2]$ -algebra when n is odd.)

It is shown by a concrete equation-counting argument in [Con14, Thm. C.1.5] that if $n = 2m$ with $m \geq 1$ then the orthogonal group scheme O_n is $\mathbf{Z}$ -smooth and in fact (see also [Con14, (C.2.2), Prop. C.2.8, Def. C.2.10, Thm. C.2.11]) is an extension of the constant group $\mathbf{Z}/(2)$ by a smooth affine group scheme SO_n with connected fibers of dimension $n(n-1)/2$ (this latter group scheme is *not* defined to be $O_n \cap \mathrm{SL}_n$ because for even n this gives the wrong group in characteristic 2; the Dickson morphism defined in (C.2.2) of loc. cit. gives a unified definition over $\mathbf{Z}$ for even n , recovering the usual notion over $\mathbf{Z}[1/2]$ by [Con14, Cor. C.3.2]).

In [Con14, Rem. C.2.12], it is shown that if $n = 2m + 1$ then $O_n = \mu_2 \times \mathrm{SO}_n$ as $\mathbf{Z}$ -group schemes for the $\mathbf{Z}$ -group scheme $\mathrm{SO}_n := O_n \cap \mathrm{SL}_n$ that is smooth with connected fibers of dimension $n(n-1)/2$ [Con14, Thm. C.2.11]. Thus, over $\mathbf{Z}[1/2]$ we recover the desired smoothness for O_n for odd n !

REMARK D.2.1. The description $O_{2m+1} = \mu_2 \times \mathrm{SO}_{2m+1}$ over $\mathbf{Z}$ allows us to fill in the loose end in Remark D.1.1 concerning the necessity of u being a square (for those who didn’t completely ignore Remark D.1.1, which we don’t logically need anyway). Namely, the Isom-scheme $I = \mathrm{Isom}(Q', Q)$ that we want to have an A -point is a left torsor for $O(Q) = O_{2m+1} = \mu_2 \times \mathrm{SO}_{2m+1}$ for the fppf topology on A . To get from Q to Q' over the fppf cover $A' = A[T]/(T^2 - u)$ of A , we can apply the diagonal operation on A'^m given by

$$\mathrm{diag}(\sqrt{u}, \sqrt{u}, 1/\sqrt{u}, \sqrt{u}, 1/\sqrt{u}, \dots, \dots, \sqrt{u}, 1/\sqrt{u}).$$

Thus, the resulting fppf 1-cocycle over $A' \otimes_A A'$ is given by

$$\mathrm{diag}(\zeta, \zeta, 1/\zeta, \dots, \zeta, 1/\zeta)$$

for $\zeta = \sqrt{u} \otimes (1/\sqrt{u})$. But $\zeta^2 = u \otimes (1/u) = 1$, so $\zeta \in \mu_2(A' \otimes_A A')$ and hence $1/\zeta = \zeta$.

We conclude that the fppf descent datum describing I as an O_{2m+1} -torsor comes from $H^1(A, \mu_2)$. Since $O_{2m+1} = \mu_2 \times SO_{2m+1}$, this O_{2m+1} -torsor is trivial if and only if the class in $H^1(A, \mu_2)$ is trivial (since a 1-coboundary splitting the O_{2m+1} -valued 1-cocycle over some fppf cover of A refining A' can be projected to its μ_2 -factor). By fppf Kummer theory, the class in $H^1(A, \mu_2)$ represented by $\sqrt{u} \otimes 1/\sqrt{u} \in (A' \otimes_A A')^\times$ corresponds to the class of $u \in A^\times/(A^\times)^2$ under the natural injective map $A^\times/(A^\times)^2 \rightarrow H^1(A, \mu_2)$. Hence, $I(A)$ is non-empty if and only if u is a square in A . This is exactly the necessity claimed in Remark [D.1.1](#).

Appendix E

From normal crossings to strict normal crossings

§E.1 Introduction

Let S be an integral noetherian scheme. Recall that we defined an effective Cartier divisor $D \subset S$ to be a *strict normal crossings divisor* if S is regular at each point $d \in D$ and the irreducible components $\{D_i\}_{i \in I}$ of D (with their reduced structure) are regular and have (scheme-theoretic) overlaps $D_J = \cap_{i \in J} D_i$ that are regular with pure codimension $\#J$ for all subsets J of I . (The case $J = \emptyset$ is either a tautology or is to be ignored; take your pick.) Locally, this says that for each $d \in D$ lying on exactly m irreducible components, a local generator $f \in \mathcal{O}_{S,d}$ for the ideal sheaf of D at d has irreducible factorization $f = \prod_i f_i$ in $\mathcal{O}_{S,d}$ with the f_i 's pairwise coprime (that is, not unit multiples of each other) and part of a regular system of parameters of $\mathcal{O}_{S,d}$.

We also defined the concept of *normal crossings divisor*: a closed subscheme $D \subset S$ is a normal crossings divisor if it pulls back to a strict normal crossings divisor on an étale cover of S (or equivalently, for each $s \in S$ there is an étale map $U \rightarrow S$ hitting s so that the pullback of D in U is a strict normal crossings divisor in U). In particular, D must be reduced and have invertible ideal sheaf (that is, it is Cartier). The purpose of this appendix is to prove that for any normal crossings divisor $D \subset S$ there is an integral noetherian scheme S' and a proper birational map $\varphi : S' \rightarrow S$ that is an isomorphism outside of D (and in fact is a blow-up along a coherent ideal whose zero scheme is physically supported in D , but we do not require this fact) such that $\varphi^{-1}(D)_{\text{red}}$ is a strict normal crossings divisor on S' . The reason we wish to prove this result is two-fold: it is a very basic fact for which we do not know a reference, and at the end of deJong's proof he gets a divisor that is merely a normal crossings divisor (due to a local calculation in the étale topology) so we need such a φ to get the “strict normal crossings” aspect of deJong's theorem.

REMARK E.1.1. Note that if the original S is regular then so is S' obtained in this way, since by design S' is regular at points in $\varphi^{-1}(D)$ (by the sncd condition) whereas $S' - \varphi^{-1}(D) \rightarrow S - D$ is an isomorphism (so $S' - \varphi^{-1}(D)$ inherits regularity from S). This is important for the final resolution process in deJong's proof.

§E.2 Construction of some closed sets

At the end of [deJ96, §2.4] it is explained how one carries out the construction of $\varphi : S' \rightarrow S$ as above. The explanation involves the étale-local notion of “branches”, so to make contact with the original situation one really wants a global intrinsic (rather than étale-local) description of the process and for a general noetherian integral scheme (as opposed to a variety over a field) in order to carry out the relevant étale descent. Hence, one may pause to question if everything really works

out correctly.

We prove here that everything does indeed work as it should. Given a normal crossings divisor $D \subset S$, our goal is to construct a finite sequence of blowings-up

$$\varphi : S_n \rightarrow S_{n-1} \rightarrow \cdots \rightarrow S$$

along closed subschemes supported over D such that $\varphi^{-1}(D)_{\text{red}} \subset S_n$ is a strict normal crossings divisor. It then follows from the self-contained (but slightly obscurely proved) [RG71, Lemma 5.14] that such a composite map φ is necessarily a blow-up along a closed subscheme physically supported in D , though this fact that φ is actually a blow-up within D (rather than merely an isomorphism over $S - D$) is not necessary for deJong's work.

For $n \geq 1$, let $Z_n \subset D$ be the locus where D has “ $\geq n$ analytic branches”. Rigorously,

$$Z_n = \{d \in D \mid \mathcal{O}_{D,d}^{\text{sh}} \text{ has } \geq n \text{ minimal primes}\}.$$

This is merely a subset of D , and its geometry will be sorted out in a moment. Observe that the formation of Z_n is compatible with étale localization on S . That is, if $S' \rightarrow S$ is étale and if D' is the pullback of D to S' and $Z'_n \subset D'$ is defined analogously to $Z_n \subset D$, then Z'_n is the full preimage of Z_n in S' .

The following lemma isolates some intrinsic properties of strict normal crossings divisors that are local for the étale topology and so are well-posed for normal crossings divisors such as our D :

LEMMA E.2.1. Let $D \subset S$ and $\{Z_n\}_{n \geq 1}$ be as above.

1. The scheme S is regular at all points of D .
2. The subset $Z_n \subset S$ is Zariski-closed with pure codimension n if it is nonempty; that is, $\inf_{z \in Z_n} \dim \mathcal{O}_{S,z} = n$ if $Z_n \neq \emptyset$.
3. Giving Z_n its reduced structure, the regular locus Z_n^{reg} is equal to the open subset $Z_n - Z_{n+1}$ in Z_n , and this is both dense in Z_n as well as of pure codimension n in S .

Proof. For (1) we may work étale-locally, so the assertion is clear (as it holds for strict normal crossings divisors). As for (2), since the formation of the set Z_n is local for the étale topology and since étale descent theory is effective for descent of reduced closed subschemes it suffices to prove the result étale-locally over S . Hence, for (2) it is enough to treat the case when D is a strict normal crossings divisor. By (1) and [EGA, IV₄, 5.1.9], it likewise follows that for the proof of (3) it is enough to check it étale-locally on S . Thus, to prove both (2) and (3) we lose no generality in assuming that D is a strict normal crossings divisor in S . Since the Cartier D is reduced, if $\{D_i\}$ is the set of (reduced) irreducible components of D then $D = \sum D_i$ as closed subschemes of S . By the regularity conditions in the definition of a strict normal crossings divisor, for all $d \in D$

the irreducible components of $\text{Spec } \mathcal{O}_{D,d}^{\text{sh}}$ are given by $\text{Spec } \mathcal{O}_{D_i,d}^{\text{sh}}$ for D_i that contain d . Hence, $Z_n = \cup_{\#J=n} D_J$ for all $n \geq 1$, so Z_n is Zariski-closed. This settles (2).

The open complement $Z_n - Z_{n+1}$ in Z_n with its reduced structure is given by

$$Z_n - Z_{n+1} = \coprod_{\#J=n} (D_J - \cup_{i \notin J} D_i),$$

and this is regular with pure codimension n in S if it is nonempty because each $D_J - \cup_{i \notin J} D_i$ is regular with pure codimension n in S if it is nonempty. Hence, $Z_n - Z_{n+1} \subseteq Z_n^{\text{reg}}$. To get the reverse inclusion, pick $d \in Z_n$ and suppose $d \in D_J$ with $\#J = n$. Hence, $\text{Spec } \mathcal{O}_{D_J,d} \subset \text{Spec } \mathcal{O}_{S,d}$ is a closed subscheme of codimension n that is regular. But $Z_n = \cup_{\#J=n} D_J$ set-theoretically, so $\text{Spec } \mathcal{O}_{Z_n,d}$ is set-theoretically the union of the $\text{Spec } \mathcal{O}_{D_J,d}$'s for all D_J that contain d (with $\#J = n$) and these $\text{Spec } \mathcal{O}_{D_J,d}$'s *must* be the irreducible components of $\text{Spec } \mathcal{O}_{Z_n,d}$ (as they are irreducible and none contains the other since they all have the same dimension). Hence, if $\text{Spec } \mathcal{O}_{Z_n,d}$ is regular then d must lie in a *unique* D_J with $\#J = n$. This also gives the density of $Z_n - Z_{n+1}$ in Z_n . $\square$

§E.3 The blow-up process

We are now ready to make some blow-ups. Let m be maximal so that Z_m is nonempty, and consider the strictly increasing chain of closed sets

$$\emptyset \neq Z_m \subsetneq Z_{m-1} \subsetneq \cdots \subsetneq Z_1 = D.$$

By Lemma E.2.1, each $Z_i \subset S$ is of pure codimension i , the open locus $Z_i - Z_{i-1} \subset Z_i$ is dense and equal to Z_i^{reg} , and Z_m is regular. If $m = 1$ then D is regular and we are done. Hence, we shall assume $m > 1$, and the first blow-up we shall study is that of S along Z_m .

Let $S' = \text{Bl}_{Z_m}(S)$. We claim that the reduced preimage D' of D in S' is a normal crossings divisor equal to $D' = \widetilde{D} \cup E$ as closed sets with $\widetilde{D}$ the strict transform of D and E the exceptional divisor of the blow-up; moreover, E is *regular*. To justify this, by working locally for the étale topology we come down to the easily checked fact that in a regular local ring A with $\{f_1, \dots, f_m\}$ part of a regular system of parameters, in the regular *integral* scheme

$$\text{Spec } A[t_1, \dots, t_n]/(f_j - f_1 t_j)$$

the locally closed subscheme $\{f_{j_0} \neq 0, f_1 = 0\}$ for $j_0 \neq 1$ is empty (yet in $\text{Spec } A$ the locus with the same description is dense in $\{f_1 = 0\}$) whereas the locally closed subscheme $\{f_{j_0} = 0, f_1 \neq 0\}$ in this integral scheme has closure equal to the *regular* Cartier divisor $\{t_{j_0} = 0\}$.

It follows that after some étale local base change on S (to make the normal crossings divisor D become a strict normal crossings divisor) the m -fold overlaps of strict transforms of irreducible components of D are empty, so the strict transform $\widetilde{Z}_{m-1}$ of Z_{m-1} is a *disjoint* union of $(m-1)$ -

fold overlaps of such strict transforms étale locally on S . Hence, $\tilde{Z}_{m-1}$ is *regular*. Again working étale locally over S to make D a strict normal crossings divisor, on the blow-up S' the $(m+1)$ -fold overlaps among irreducible components of D' are empty because they involve at most one irreducible component of the regular exceptional divisor and so at least m of the strict transforms of the irreducible components of D (and we have just seen that such m -fold overlaps in the blow-up are empty).

Now consider the blow-up $S'' = \text{Bl}_{\tilde{Z}_{m-1}}(S')$. If we work Zariski-locally on S' and use the disjoint union decomposition of $\tilde{Z}_{m-1}$ into certain $(m-1)$ -fold overlaps, the same calculations as above show:

- the reduced preimage $D'' \subset S''$ of D' in S' is a normal crossings divisor;
- if $m > 2$, the strict transforms $\tilde{Z}_{m-2}$ of Z_{m-2} with respect to $S'' \rightarrow S'$ is regular. (The key point is that $Z_{m-2} - Z_{m-1}$ is a dense open in Z_{m-2} if $m > 2$.)

We can keep iterating this procedure until we arrive at a (composite) blow-up $\bar{S} \rightarrow S$ that is an isomorphism over $S - D$ and on which the reduced preimage $\bar{D}$ of D is a normal crossings divisor. In fact, we can arrange that $\bar{D}$ is the union of some regular irreducible components and the strict transform $\tilde{D}$ of D in $\bar{S}$, with the strict transform $\tilde{Z}_1$ of Z_1 in $\bar{S}$ also regular. But $\tilde{Z}_1 = \tilde{D}$ by construction, so we conclude that $\tilde{D}$ is a *regular* normal crossings divisor, so its irreducible components are its connected components and thus are regular. Thus, $\bar{D} \subset \bar{S}$ is a normal crossings divisor whose irreducible components are regular, so $\bar{D}$ is a strict normal crossings divisor in $\bar{S}$.

Appendix F

Resolving singularities in codimension 3

§F.1 Axiomatic setup

This appendix gives a detailed treatment of the final two sections [deJ96, §4.26-4.27] of the proof of the main result over fields. We begin by fixing any pair (X, Z) satisfying the hypotheses at the end of [deJ96, §4.25]: X is a projective variety of dimension $d \geq 3$ over an algebraically closed field k and $Z \subset X$ is a reduced closed subset such that:

1. $Z \cap X^{\text{sm}}$ is a normal crossings divisor in X^{sm} (recall this means that locally for the étale topology on X^{sm} the closed subset looks like an intersection of coordinate hyperplanes through the origin in an affine space),
2. for all $x \in (Z \cap \text{sing}(X))(k)$ there is a k -algebra isomorphism

$$\hat{\mathcal{O}}_{X,x} \simeq k[[u, v, t_1, \dots, t_{d-1}]]/(uv - t_1 \cdots t_s)$$

with the quotient $\hat{\mathcal{O}}_{X,x}$ cut out by the principal ideal generated by $t_1 \cdots t_r$ where $2 \leq s \leq r \leq d-1$ (and r, s may depend on x),

3. all irreducible components E of $\text{sing}(X)$ are smooth of dimension $d-3$ and all overlaps $E \cap E'$ (in the scheme-theoretic sense) are smooth (possibly not transverse intersections).

These hypotheses imply that the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is invertible (as we may check locally at each $x \in Z(k)$ by separately treating the cases $x \in X^{\text{sm}}(k)$ and $x \in \text{sing}(X)(k)$), so Z has pure codimension 1 in X .

Note that if X is smooth then by (1) we have that Z is a normal crossings divisor in X . Hence, if X is smooth then we may apply Appendix E on passage from normal crossings divisors to *strict* normal crossings divisors (via precise blow-ups) to find a modification $X' \rightarrow X$ with *smooth* X' (see Remark E.1.1!) such that the preimage of Z in X' is a strict normal crossings divisor (i.e., the irreducible components of Z are smooth and are mutually transverse). This final situation is exactly where we want to get to complete the inductive argument, so we may now assume $\text{sing}(X)$ is non-empty.

We wish to find a modification $X' \rightarrow X$ such that for the preimage Z' of Z in X' we have that (X', Z') satisfies the same axioms as above for the pair (X, Z) but with $\text{sing}(X')$ somehow “simpler” than $\text{sing}(X)$ so that the process eventually ends with no singularities in the total space (and then a further modification of X via Appendix E brings us to the desired case, namely with Z a strict normal crossings divisor in the smooth X).

§F.2 Blow-up in codimension 3

Let E be an irreducible component of $\text{sing}(X)$, and let $\pi : X' = \text{Bl}_E(X) \rightarrow X$ be the blow-up. Let $Z' = \pi^{-1}(Z)_{\text{red}}$. The answer to our prayers is given by the following result from [deJ96, §4.27]:

THEOREM F.2.1. With notation as just defined, the pair (X', Z') satisfies the same axioms as (X, Z) in §F.1, but the number of irreducible components of $\text{sing}(X')$ is one less than the number of irreducible components of $\text{sing}(X)$. More precisely, the irreducible components of $\text{sing}(X')$ are the strict transforms of the irreducible components of $\text{sing}(X)$ distinct from E .

Proof. The idea is similar to how we analyzed (and ultimately removed the presence of) codimension-2 irreducible components in $\text{sing}(X)$: we do some concrete blow-up calculations in convenient étale charts. As before, we need to exercise some care with respect to keeping track of irreducibility under completion and étale localization, and once again regularity will be the key to handling such issues.

Let $\tilde{E}$ be an irreducible component of $\text{sing}(X)$ distinct from E (if any exist). There is a natural E -isomorphism between $\text{Bl}_{E \cap \tilde{E}}(E)$ and the strict transform $\tilde{E}'$ of $\tilde{E}$ under the blow-up $\pi : X' \rightarrow X$ along E (where $E \cap \tilde{E}$ is taken in the scheme-theoretic sense, which is regular by our axioms). This relationship between strict transform of a closed subset under a blow-up and a blow-up of the closed subset is explained in detail in Lemma A.3.3. The advantage of such a “blow-up” description of $\tilde{E}'$ in X' is that it shows that this strict transform (which we know to be integral: it is a closure of $\tilde{E} \cap (X - E)$ in X') is *regular*. Indeed, working locally for the étale topology, the study of $\text{Bl}_{E \cap \tilde{E}}(E)$ is the same as that of blowing up an affine space along a linear subspace, for which the blow-up is seen to be regular by explicit calculation on blow-up charts.

The upshot is this: all such $\tilde{E}'$ ’s are *smooth* codimension-3 subvarieties of X' . Since $X' - \pi^{-1}(E) \simeq X - E$, all singularities of X' away from the $\tilde{E}'$ ’s must lie in the exceptional divisor. Thus, our problem is to verify the following three assertions, with $\{E_i\}$ the set of irreducible components of $\text{sing}(X)$:

1. X' is regular at all points of along $\pi^{-1}(E)$ away the reduced union $\cup_{E_i \neq E} E'_i$ (this is the strict transform of the closure of $\text{sing}(X) - E$, even when this complement is empty!),
2. for all $E_i, E_j \neq E$ with $i \neq j$ (if two such exist!), the overlap $E'_i \cap E'_j$ in the scheme-theoretic sense is smooth (or equivalently, regular),
3. At all k -points of $Z' \cap \pi^{-1}(E)$ the formal structure of the pair (X', Z') is as in the second axiom at the outset.

By the same type of zig-zag argument through étale neighborhoods (now around a k -point of X) via Artin approximation, it is enough (check!) to study the analogous situation for an étale chart with the same formal structure as (X, x) (i.e., same completed local ring).

Our problem is now reduced to checking the above three assertions for the following “open variety” situation:

$$X = \operatorname{Spec} k[u, v, t_1, \dots, t_{d-1}] / (uv - t_1 \cdots t_s), \quad Z = V(t_1 \cdots t_r) \hookrightarrow X$$

with $2 \leq s \leq r \leq d-1$ and $E = V(u, v, t_i, t_j)$ with $1 \leq i < j \leq s$ (where $V(I)$ denotes the zero-scheme in X of an ideal I). Without loss of generality, $i = 1$ and $j = 2$. The blow-up $\operatorname{Bl}_E(X)$ is covered by four Zariski-open subschemes, namely the charts $D_+(u)$, $D_+(v)$, $D_+(t_1)$, and $D_+(t_2)$. By symmetry, it suffices to study the opens $D_+(u)$ and $D_+(t_1)$. We will see by calculation that $D_+(u)$ is regular with $Z' \cap D_+(u)$ a strict normal crossings divisor (which is only enough to infer the normal crossings condition in the original situation, as we are presently only working on an étale chart of the original setup and the strict normal crossings condition is not local for the étale topology whereas the normal crossings condition is). In particular, all of the real work will therefore take place in $D_+(t_1)$. Keep in mind, as we have already seen, that $Z' = \pi^{-1}(Z)_{\operatorname{red}}$.

On the open locus $D_+(u)$ we have relations $v = v'u$, $t_1 = t'_1 u$, and $t_2 = t'_2 u$, so the relation $uv = t_1 \cdots t_s$ is the same as $u^2(v' - t'_1 t'_2 \cdots t_s) = 0$. Killing the u -power torsion therefore gives that that

$$D_+(u) = \operatorname{Spec}(k[u, t'_1, t'_2, t_3, \dots, t_{d-1}])$$

since this clearly has no nonzero u -power torsion. Note by inspection that this is regular. Also, $t_1 \cdots t_r = u^2 t'_1 t'_2 t_3 \cdots t_r$, so since u is not a zero-divisor on $D_+(u)$ we conclude that the overlap $Z' \cap D_+(u) = \pi^{-1}(Z)_{\operatorname{red}} \cap D_+(u)$ is the integral zero-scheme cut out by $ut'_1 t'_2 t_3 \cdots t_r = 0$. By inspection, $Z' \cap D_+(u)$ is clearly a strict normal crossings divisor in $D_+(u)$.

Now we turn our attention to $D_+(t_1)$, which is a bit more involved. We have the relations $u = u't_1$, $v = v't_1$, and $t_2 = t'_2 t_1$, so killing t_1^2 -torsion gives

$$D_+(t_1) = \operatorname{Spec}(k[u', v', t_1, t'_2, \dots, t_{d-1}] / (u'v' - t'_2 t_3 \cdots t_s)), \quad Z' = V(t_1 t'_2 t_3 \cdots t_r)$$

by the same sort of calculation as on $D_+(u)$. Note that the relation cutting out $D_+(t_1)$ does not involve t_1 . Thus, the singular locus on $D_+(t_1)$ is the union of the irreducible loci of the following two types:

$$V(u', v', t'_2, t_i), \quad V(u', v', t_i, t_j) \tag{F.2.1}$$

with $3 \leq i \leq s$ and $3 \leq i < j \leq s$ respectively. These are clearly smooth and have smooth scheme-theoretic overlaps. By direct inspection we see that $Z' \cap D_+(t_1)$ has the desired formal structure within $D_+(t_1)$ at all of its k -points lying in $\operatorname{sing}(X')$.

Our final task is to relate the irreducibles in (F.2.1) to the strict transforms of the irreducible components of $\operatorname{sing}(X)$ distinct from $E = V(u, v, t_1, t_2)$. This is purely a counting problem and only requires us to work generically on irreducible components of singular locus on X and X' (so it is enough to identify work out how these meet either of $D_+(t_1)$ or $D_+(t_2)$; there is no

need to work on $D_+(u)$ or $D_+(v)$ because we have seen that X' is smooth on these latter two opens). By consideration of generic points, the map $D_+(t_1) \subset X' \rightarrow X$ induces a dominant map $V(u', v', t'_2, t_i) \rightarrow V(u, v, t_2, t_i)$ for $3 \leq i \leq s$, so the closed locus $V(u', v', t'_2, t_i)$ in $D_+(t_1)$ is where the strict transform of $V(u, v, t_2, t_i)$ meets $D_+(t_1)$. Likewise, for $3 \leq i < j \leq s$ we get a dominant map $V(u', v', t_i, t_j) \rightarrow V(u, v, t_i, t_j)$ that realizes the source as the part of $D_+(t_1)$ meeting the strict transform of the target. There remain the irreducible components $V(u, v, t_1, t_i)$ of $\text{sing}(X)$ with $3 \leq i \leq s$, but since t_2 is a *unit* at the generic point of each of these components we see that these generic points must all lift into $D_+(t_2)$ under $X' \rightarrow X$. Hence, we may analyze these strict transforms generically by working in $D_+(t_2)$ exactly as we just worked out where $D_+(t_1)$ meets a dense open in the strict transform of $V(u', v', t'_2, t_i)$ (swap the roles of t_1 and t_2). $\square$

References

- [Art69] M. Artin. “Algebraization of formal moduli I”. In: *Global analysis* (1969), pp. 21–71.
- [Art74] M. Artin. “Versal deformations and algebraic stacks”. In: *Inventiones Mathematicae* 27 (1974), pp. 165–189.
- [SP] Stacks Project Authors. *Stacks Project*. URL: <https://stacks.math.columbia.edu>.
- [Ber93] V. Berkovich. “Étale cohomology for non-archimedean analytic spaces”. In: *Publ. Math. IHES* 78 (1993), pp. 5–161.
- [BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron Models*. Springer-Verlag, 1990.
- [Con] B. Conrad. *Algebraic Groups I*. URL: <https://www.ams.org/open-math-notes/omn-view-listing?listingId=110662>.
- [Con02] B. Conrad. “A modern proof of Chevalley’s theorem on algebraic groups”. In: *Journal of the Ramanujan Mathematical Society* 17.1 (2002), pp. 1–18.
- [Con14] B. Conrad. “Reductive Group Schemes”. In: *Panoramas et Synthèses* 42-43 (2014), pp. 93–444.
- [Con15] B. Conrad. *Minimal models for elliptic curves*. 2015. URL: <https://math.stanford.edu/~conrad/papers/minimalmodel.pdf>.
- [CS86] G. Cornell and J. Silverman. *Arithmetic Geometry*. Springer-Verlag, 1986.
- [deJ96] A.J. deJong. “Smoothness, semi-stability, and alterations”. In: *Publ. Math. IHES* 83 (1996), pp. 51–93.
- [Ad97] Dan Abramovich and A.J. deJong. “Smoothness, semi-stability, and toroidal geometry”. In: *Journal of Algebraic Geometry* 6 (1997), pp. 789–801.
- [SGA7] P. Deligne and N. Katz. *Groupes de monodromie en géométrie algébrique II*. Lecture Notes in Math. Springer-Verlag, 1973.
- [DM69] P. Deligne and D. Mumford. “The irreducibility of the space of curves of given genus”. In: *Publ. Math. IHES* 36 (1969), pp. 75–109.
- [DR73] P. Deligne and M. Rapoport. “Les schémas de modules des courbes elliptiques”. In: *Lecture Notes in Math* 349 (1973), pp. 143–316.
- [FK88] E. Freitag and R. Kiehl. *Étale Cohomology and the Weil Conjecture*. Springer-Verlag, 1988.
- [FL85] W. Fulton and S. Lang. *Riemann-Roch Algebra*. Springer-Verlag, 1985.
- [Gro60] A. Grothendieck. “Techniques de construction en géométrie analytique V. Fibrés vectoriels, fibrés projectifs, fibrés en drapeaux”. In: *Séminaire Cartan Vol.13* (1960-61), pp. 1–15. URL: <https://www.numdam.org/actas/SHC/>.

- [EGA] A. Grothendieck. “Eléments de Géométrie Algébrique”. In: *Publ. Math. IHES* (4, 8, 11, 17, 20, 24, 28, 32 1960-67).
- [Gro66] A. Grothendieck. “On the deRham cohomology of algebraic varieties”. In: *Publ. Math. IHES* 29 (1966), pp. 95–103.
- [SGA1] A. Grothendieck. *Seminaire de géométrie algébrique 1*. Lecture Notes in Math. Springer-Verlag, 1971.
- [SGA4] A. Grothendieck. *Seminaire de géométrie algébrique 4*. Lecture Notes in Math. Springer-Verlag, 1972.
- [Har77] R. Hartshorne. *Algebraic Geometry*. Vol. 52. Graduate Texts in Mathematics. Springer-Verlag, 1977.
- [Hir64a] Heisuke Hironaka. “Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero: I”. In: *Annals of Mathematics* 79.1 (1964), pp. 109–203. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1970486> (visited on 09/23/2025).
- [Hir64b] Heisuke Hironaka. “Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero: II”. In: *Annals of Mathematics* 79.2 (1964), pp. 205–326. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1970547> (visited on 09/23/2025).
- [Jou83] J-P. Jouanolou. *Théorèmes d’Bertini et Applications*. Birkhäuser, 1983.
- [Knu83] F. Knudsen. “The Projectivity of the Moduli Space of Stable Curves II: The Stacks $M_{g,n}$ ”. In: *Math. Scand.* 52 (2 1983), pp. 161–199.
- [Kol07] János Kollár. *Lectures on Resolution of Singularities (AM-166)*. Princeton University Press, 2007. ISBN: 9780691129235. URL: <http://www.jstor.org/stable/j.ctt7rptq> (visited on 09/23/2025).
- [LMB] G. Laumon and L. Moret-Bailly. *Champs Algébriques*. Springer-Verlag, 2000.
- [Liu02] Q. Liu. *Algebraic Geometry and Arithmetic Curves*. Oxford University Press, 2002.
- [Mat80] H. Matsumura. *Commutative Algebra (2nd ed.)*. W.A. Benjamin Inc., 1980.
- [Mat90] H. Matsumura. *Commutative Ring Theory*. Cambridge Univ. Press, 1990.
- [Mum12] D. Mumford. *Abelian Varieties*. Tate Institute for Fundamental Research, 2012.
- [Nit05] N. Nitsure. *Fundamental Algebraic Geometry: FGA Explained*. Vol. 123. Mathematical Surveys and Monographs. American Mathematical Society, 2005.
- [Ols05] M. Olsson. “On proper coverings of Artin stacks”. In: *Advances in Mathematics* 198 (2005), pp. 93–106.

- [RG71] M. Raynaud and L. Gruson. “Crières de platitude et de projectivité. Techniques de “platification” d’un module”. In: *Inventiones Mathematicae* 13 (1971), pp. 1–89.
- [Ser79] J-P. Serre. *Local Fields*. Vol. 67. Graduate Texts in Mathematics. Springer-Verlag, 1979.
- [Swa98] R. Swan. “Néron-Popescu desingularization”. In: *Lect. Algebra Geom.* 2 (1998), pp. 135–192.