Graphs, Matroids and Polynomial Countability

(Adapted) Notes from Melody Chan's 2022 Advanced PROMYS Course

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Contents

Overview

This is a series of notes from a course originally taught by Prof. Melody Chan in PROMYS 2022 on a conjecture of Kontsevich regarding polynomiality of point-counting functions on schemes associated with the spanning-tree polynomials of connected graphs. These notes assume no background of algebraic geometry whatsoever. Any reader familiar with a first course in linear algebra and field theory[1] should be able to reasonably follow these notes. In order to achieve this, however, there naturally had to be a compromise so lots of results (particularly in the later lectures) will be handwaved away. The goal of this

[1]Prof. Chan's original course was made accessible to advanced high-school students with no background in field theory or linear algebra. For sake of brevity in adapted notes we assume this elementary material.

is to simply give interested young mathematicians an appreciation for the connections between these seemingly disparate fields, and the methods used to link them.

Lecture 1

(Finite Affine $\mathbb{Z}$-)Schemes

We begin abruptly and boldly with a definition.

DEFINITION 1.1. A **finite type affine** $\mathbb{Z}$-scheme is a finite list of integer polynomials in finitely many variables. We denote it as follows:

$$X = V(f_1, \dots, f_k).$$

If $f_1, \dots f_k \in \mathbb{Z}[x_1, \dots, x_n]$ and K is a field then we denote by $X(K)$ the **vanishing locus** of the f_i over the vector space K^n, i.e:

$$X(K) := \left\{ \vec{a} = (a_1, \dots a_n) \in K^n \ : \ f_i(\vec{a}) = 0, \ \forall i \right\}.$$

The elements of K are called the K**-rational points of** X.

ASIDE FOR EXPERTS. Readers familiar with algebraic geometry may see this as a painful oversimplification, but in our case it is far better to get across the core idea than spend six weeks on abstract nonsense.

SANITY CHECK. Suppose I took the above definition and I replaced $\mathbb{Z}$ with $\mathbb{F}_5$ for example. If I were to let X be a finite-type affine scheme over $\mathbb{F}_5$, then what would it mean to consider the $\mathbb{F}_7$-rational points of X? Check that for arbitrary fields this doesn't make much sense. Why can we do it for $\mathbb{Z}$?.

EXAMPLE 1.2 (The Circle). Consider the finite type affine scheme defined by the single equation $f(x, y) = x^2 + y^2 - 1$. Then:

- We can visualize $X(\mathbb{R})$ as the standard unit circle in the Cartesian plane.
- $X(\mathbb{Q})$ would correspond to points on the unit circle with rational coefficients. The fact that there are infinitely many of these comes from the fact that we can generate infinitely many Pythagorean triples (that aren't multiples of each other).
- It is not too hard to verify that:

$$X(\mathbb{F}_7) = \{(0, \pm 1), (\pm 1, 0), (\pm 2, \pm 2)\}$$

We now introduce what it means for a scheme to be *Polynomially Countable*, a central concept of this course. Some notation first:

- We define $\mathcal{Q}$ to be the set of all prime powers, i.e $\{2, 3, 4, 5, 7, 9, \ldots\}$.
- For a scheme X, we define a function:

$$\begin{aligned} |X| : \mathcal{Q} &\to \mathbb{Z}_{\geq 0} \\ q &\mapsto \#X(\mathbb{F}_q) \left(:= |X(\mathbb{F}_q)| \right) \end{aligned} \tag{1}$$

  that maps each prime power q to the number of $\mathbb{F}_q$-rational points of X. We can refer to this as the **point-counting function** of X.

Warning (Non-ideal Notation). In the context of this text, $|X|$ for X a scheme will always refer to the funcction defined above. Otherwise, $|S|$ for some arbitary set or object (that ISN'T a scheme) will refer to the cardinality of S. So $|X|$ is a function and $|X(K)|$ is a cardinality of $X(K)$.

Example 1.3. Let X be the scheme defined by the single polynomial $x_1 x_2 \cdots x_n \in \mathbb{Z}[x_1, \ldots, x_n]$. In order for this polynomial to vanish at a point $\vec{a} = (a_1, \ldots, a_n) \in K^n$, at least one of the coordinates a_i must be zero. Using a complementary counting argument we thus see that

$$|X|(q) = q^n - (q-1)^n,$$

where the positive term in the sum counts the number of points in $\mathbb{F}_q^n$ and the negative term counts the number of points where none of the coordinates are 0.

Definition 1.4. A scheme X is **polynomially countable** if the function $|X|$ is a polynomial, i.e there is some $\phi \in \mathbb{Q}[t]$ such that $\phi(q) = |X|(q)$ for all $q \in \mathcal{Q}$.

An example of a polynomially countable scheme is that in 1.3. Not every scheme is so, in fact there are very simple counterexamples.

Non-Example 1.5 (A NON-polynomially countable scheme). Consider the scheme defined by the single equation $3x \in \mathbb{Z}[x]$. Then:

$$|X|(q) = \begin{cases} 1, & 3 \nmid q \text{ (i.e } \mathbb{F}_q \text{ has characteristic not 3)} \\ q, & 3 \mid q \text{ (i.e } \mathbb{F}_q \text{ has characteristic 3)} \end{cases}$$

which is not a polynomial.

The more general notion of a **scheme** is a bit too involved for us to dive into here, and none of the material we cover will require knowing it in full generality. For our purposes here, we can think of schemes as an object that "has points" over various fields. For example, $\mathbb{Z}$-polynomials are easy to define as schemes since we can think of its points as solutions to that polynomial on vector spaces over various fields.

Example 1.6 (Matrix Schemes).

- The GL_n scheme: Consider the object defined by "invertible $n \times n$ matrices", more commonly known as the general linear group. This, too, takes points over various fields K, though the geometry is a bit less straightforward than in the polynomial case. We can think of $GL_n(K)$ as being some set of points in K^{n^2}, each vector corresponding to an invertible matrix.
- The Sym_r^n scheme: In a similar vein to the previous example, consider the object defined by "symmetric $n \times n$ matrices of rank r". Again, we can think of this geometrically as some subset of K^{n^2}.
- Similarly you could think of all the standard matrix groups – SO, SU, SL, etc – as schemes.

Note that we haven't verified whether either of the examples above is a finite-type affine $\mathbb{Z}$-scheme specifically, as per Definition 1.1. We may or may not answer that later.

Lecture 2

(Multi)Graphs and Kontsevich's Conjecture

DEFINITION 2.1. A **graph** G is a finite set $V(G)$ of **vertices** and a set of **edges** $E = E(G)$ of unordered pairs of $V(G)$. Pictorially the vertices are a set of points and the edges are lines connecting pairs of points.

EXAMPLE 2.2. The graph represented by the vertex set $V(G) = \{1, 2, 3, 4\}$ and the edge set $E(G) = \{\{1, 2\}, \{1, 4\}, \{2, 3\}\}$ is pictorially represented by Figure 1.

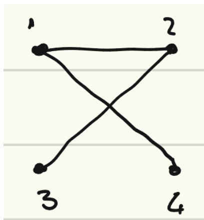

Figure 1: A pictorial representation of a graph.

The graph in Figure 1 is an example of what's called a *simple graph*, that is, a graphs without loops or multiple edges between vertices. For our purposes we might want to consider such graphs as well, which are called *multigraphs*. An example is shown in Figure 2

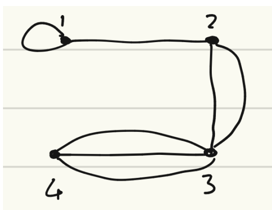

Figure 2: Example of a multigraph.

DEFINITION 2.3. A **multigraph** consists of:

(1) A finite set V, the vertex set,

(2) A finite set H, called the set of **half-edges**,

(3) A map $r : H \to V$, called the **root map**,

(4) A fixed-point-free involution $i : H \to H$ (i.e $i \circ i(h) = h$ and $i(h) \neq h$ for all $h \in H$).

We define $E = \{\{h, i(h)\} : h \in H\}$ (i.e the set of unordered pairs matched by the involution map) as the set of **edges**.

The root map tells us where each half-edge is grounded, and the fixed-point-free involution is essentially a map that associates each half-edge with its other half, so to speak.

Example 2.4 (Explicit Multigraph Description). The multigraph in Figure 2 would be represented by:

- $V = \{1, 2, 3, 4\}$.
- $H = \left\{h_{1,1',1}, h_{1',1,2}, h_{2,3,1}, h_{3,2,1}h_{2,3,2}, h_{3,2,2}, h_{3,4,1}, h_{4,3,1}, h_{3,4,2}, h_{4,3,2}, h_{3,4,3}, h_{4,3,3}\right\}$.
- $r : h_{i,j,k} \mapsto i \in V$.
- $i : h_{i,j,k} \mapsto h_{j,i,k}$.

Remark. You might wonder that the definition of a multigrap given here seems a bit unnecessarily complex. Perhaps a more intuitive one would be to use the same definition of a graph $G = (V(G), E(G))$ and simply assign a function $E(G) \to \mathbb{Z}_{>0}$ indicating the multiplicity of each edge, so to speak. In fact this is the way many sources define it.
However, as mathematicians we want to build definitions that allow us to capture and explore underlying symmetrical structure, and the use of these half-edges and involution functions may allow us to more cleanly denote automorphic structures on these graphs.

Example 2.5 (Simple Graph Examples). In the following examples we denote the vertex set $V(G)$ of the graph by $\{v_1, \dots, v_n\}$.

- If $E(G)$ is every possible pair of vertices then we denote this graph by K_n, the **complete graph** on n vertices.
- If $E(G) = \{\{v_i, v_{i+1}\} : i \in \mathbb{Z}/n\mathbb{Z}\}$ then we denote this graph by C_n the **cyclic graph** on n vertices. Note this is only a simple graph if $n \geq 3$.

Definition 2.6 (Trees). A simple graph G is **connected** if every pair of vertices has a *path* of edges connecting them. A graph is **acyclic** if it does not contain a cycle C_k as a subgraph for any k. A **tree** is a graph that is both connected and acyclic. A **leaf** is a degree-1 vertex of a tree.

Observation. Some facts about trees we don't prove but leave as exercises:

- Every connected graph must contain a tree as a subgraph.

- Every tree has a leaf.
- Every tree on n vertices has $n - 1$ edges.

Definition 2.7. For G a connected graph, a **spanning tree** T on G is a tree whose vertex set is $V(G)$ and whose edge set is a subset of $E(G)$. We denote by $T(G)$ the (nonempty!) set of all spanning trees of G.
The **spanning tree polynomial** $Q_G \in \mathbb{Z}[x_e : e \in E(G)]$ is defined as:

$$Q_G = \sum_{T \in T(G)} \prod_{e \in E(T)} x_e.$$

Example 2.8. For $G = K_3 = C_3$ (i.e a triangle), if we label the edges as e, f, g then any two pairs of edges form a spanning tree, then its spanning tree polynomial is:

$$Q_G(x_e, x_f, x_g) = x_e x_f + x_e x_g + x_f x_g \in \mathbb{Z}[x_e, x_f, x_g]$$

Observation. For any graph G with n vertices, Q_G is a homogenous polynomial of degree $n - 1$.

Conjecture (Kontsevich, '97). For any connected graph G, the scheme $V(Q_G)$ associated to the spanning tree polynomial is polynomially countable.

Remark. The original formulation stated this conjecture for *any* graph G, not just connected ones, but since any non-connected graph has no spanning trees its polynomial is 0 (or 1, or whatever), so we avoid this trivial case in our formulation.

We won't say whether the conjecture is false or true here – that is a boring binary result (that we will still eventually get to). What is far more interesting is the theory and concepts that we will explore to see the eventual resolution of this conjecture.
Our approach will initially seem a bit scattered, introducing concepts and objects that seemingly have no relation to solving this problem at hand. Solving this problem, however, will be much like falling asleep (or falling in love, if you're a Green brother); slowly, then all at once.

Lecture 3

Matroids

The motivation for this concept is to generalize the notion of linear independence beyond conventional vector spaces.

DEFINITION 3.1. A matroid consists of a pair (E, I) such that:

- E is a finite set, called our **ground set**.
- $I \subseteq \wp(E)$ is a family of subsets of E (called the **independent sets**) satisfying:
  (i) $\varnothing \in I$
  (ii) If $A \in I$ and $A' \subseteq A$ then $A' \in I$.
  (iii) If $A, B \in I$ and $|B| > |A|$ then $\exists x \in B \setminus A$ such that $A \cup \{x\} \in I$.

EXAMPLE 3.2. Let $E := \{1, \ldots, n\}$, and pick some $r \in \mathbb{Z}_{\geq 0}$. Then we set:

$$I := \left\{ E' \subseteq E : \left|E'\right| \leq r \right\}.$$

Then $M = (E, I)$ is a matroid. This type of matroid is denoted $U_{r,n}$, and called a **uniform matroid**.

EXAMPLE 3.3. Take a field F and consider the finite-dimensional vector space F^r. Then picking any n vectors $v_1, \ldots, v_n \in F^r$, consider.

- $E = \{v_1, \ldots, v_n\}$
- $I = \{A \subseteq E : A \text{ is a linearly-independent set in } F^r\}$.

It is an exercise in checking definitions to see that $M = (E, I)$ is a matroid.

DEFINITION 3.4. A matroid whose ground set E is in bijection with a list of vectors V and whose set of independent sets I is in bijection with the subsets of V that are linearly independent (as in 3.3) is called **representable**. The list of vectors V is referred to as the **representation** of the matroid.

ASIDE FOR EXPERTS. Readers familiar with group representations should be feeling some semblance of familiarity here. In both cases, we take abstract algebraic structures (groups and matroids) and realize them in vector spaces to turn difficult problems into linear algebra that we can give to undergrads.

QUESTION. Is every matroid representable?

The answer turns out to be no, though it is a little less obvious than some might think to produce a counterexample (but also not as difficult as some might think).

EXERCISE 3.A. Show that any uniform matroid $U_{r,n}$ is representable. (Hint: consider a set of n vectors in $\mathbb{R}^{r+1}$).

DEFINITION 3.5. Take a matroid $M = (E, I)$. For any $A \subset E$, we define its **rank** $r(A)$ to be the order of the maximal independent set contained in A. Naturally the rank of a matroid itself would be defined as the rank of its ground set.

When thinking about matroids as a set of vectors in a vector space, the rank of that set is the dimension of their span.

LEMMA 3.6. Given $X, Y \subseteq E$, we have:

- $0 \leq r(X) \leq |X|$
- $X \subseteq Y \implies r(X) \leq r(Y)$
- $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$

Proof. Think about it □

THEOREM 3.7 (Rank Definition of Matroids). If $r : P(E) \to \mathbb{N}$ satisfying the 3 conditions in Lemma 3.6, then r is the rank function of a unique matroid.

In other words, we could have defined matroids equivalently using the notion of rank rather than independent sets directly.

Lecture 4

Representable Matroids

Consider a rank 3 representable matroid. By definition, a representation of it must be realized by a finite set of vectors in $\mathbb{F}^3$ for some field F. We can pictorially simplify their representation: first we scale all the vectors onto a common plane in $\mathbb{F}^3$, and then any subsets of vectors before that lay in a common 2-dimensional subspace will now be collinear. Thus we can visualize the matroid as a set of points on the plane.

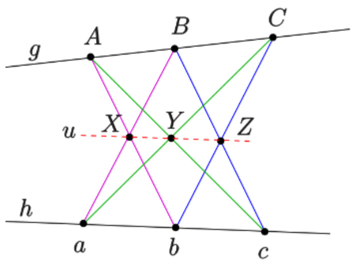

Figure 3:

NON-EXAMPLE 4.1. We define the **Non-Pappus Matroid** to be:

$$E = \{A, B, C, a, b, c, X, YZ\}$$
$$I = \left\{E' \subset E : |E| \leq 3\right\} \setminus \{\text{NON-DOTTED collinear triples of points}\}$$

One can verify that the independent sets do indeed satisfy the matroid axioms. The point is that if this matroid were representable then since $\{X, Y, Z\} \in I$ then the points X, Y, Z in Figure 3 should NOT be collinear. However, this is not possible, due to a classical theorem of Pappus.

THEOREM 4.2 (Pappus' Hexagon Theorem). Take a construction as shown in the diagram in Figure 3. The theorem states that X, Y, Z are collinear (over any field, though this part is less classical).

We omit the proof of this.

COROLLARY 4.3. The non-Pappus matroid is not representable.

To generalize the method of construction used in Example 4.1 to show nonrepresentablity, we can "draw" any rank 3 representable matroid by putting its ground set as points on the plane such that any set of non-linearly independent 3 points are collinear.

Example 4.4. Let $E = \{a, b, c, d, e, f\}$, and consider the Fano Plane as shown in Figure 4. The non-bases of I are the collinear points in the fano plane (i.e I contains any subset of size up to 3 except for collinear points), and this gives us the **Fano matroid**.

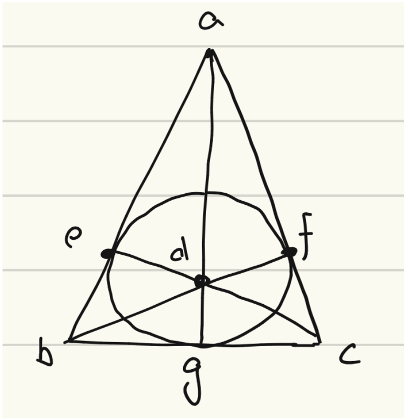

Figure 4: The Fano Matroid

Theorem 4.5. The Fano matroid is representable over $\mathbb{F}$ if and only if $\text{char}(\mathbb{F}) \neq 2$.

Proof. If $\mathbb{F}$ has characteristic 2, then we can represent the Fano plane in $\mathbb{F}^3$ by:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \text{ where the columns of the matrix are } a, b, c, d, e, f, g$$

It is not too difficult to check that sets in I are independent and sets not in I are not.

Let $\mathbb{F}$ be any field that is not characteristic 2, and suppose that $A \in \mathbb{F}^{3\times 7}$ has columns representing the matroid M. First, note that $\{a, b, c\}$ is a basis of M. Then, we may assume that:

$$A = \begin{bmatrix} 1 & 0 & 0 & | * & * & * & * \\ 0 & 1 & 0 & | * & * & * & * \\ 0 & 0 & 1 & | * & * & * & * \end{bmatrix}$$

Since $\{a, b, e\}$ is a non basis, it must hold that the last entry of e is 0. Likewise, the middle entry of f must be 0, the first entry of g must be 0. Thus,

$$A = \begin{bmatrix} 1 & 0 & 0 & | * & * & * & 0 \\ 0 & 1 & 0 & | * & * & 0 & * \\ 0 & 0 & 1 & | * & 0 & * & * \end{bmatrix}$$

Note that all the stars must be non-zero, else the linear independence conditions would fail. Moreover, we observe that if we scale any column, we can produce a matrix that still represents M. Thus, we get:

$$A = \begin{bmatrix} 1 & 0 & 0 & |1 & 1 & 1 & 0 \\ 0 & 1 & 0 & |* & * & 0 & * \\ 0 & 0 & 1 & |* & 0 & * & * \end{bmatrix}$$

Moreover, scaling the rows will also produce a matrix representing M. Thus, we get:

$$A = \begin{bmatrix} 1 & 0 & 0 & |1 & 1 & 1 & 0 \\ 0 & * & 0 & |1 & * & 0 & * \\ 0 & 0 & * & |1 & 0 & * & * \end{bmatrix}$$

Finally, scaling the columns once more we get:

$$A = \begin{bmatrix} 1 & 0 & 0 & |1 & 1 & 1 & 0 \\ 0 & 1 & 0 & |1 & w_1 & 0 & 1 \\ 0 & 0 & 1 & |1 & 0 & w_2 & w_3 \end{bmatrix}$$

Since $\{c, d, e\}$ is a non-bases, we get that we must have a linear combination $\alpha(0,0,1) + \beta(1,1,1) + \gamma(1, w_1, 0) = 0$, which tells us that $w_1 = 1$. Similarly, looking at $\{a, d, g\}$ we get $w_3 = 1$ and looking at $\{b, d, f\}$ we get that $w_2 = 1$. However, we see that $\{e, f, g\} = \{(1,1,0), (1,0,1), (0,1,1)\}$ is supposedly linearly independent which is not possible in a field except when it has characteristic 2.

□

Lecture 5

Matrix Tree Theorem

Let us restrict ourselves to simple graphs, as we are working with spanning trees.

DEFINITION 5.1. Let G be a simple graph, with $V(G) = \{v_1, \dots, v_n\}$. The **Laplacian** $L(G)$ of G is an $n \times n$ matrix with intries given by:

$$L(G)_{i,j} = \begin{cases} \#\text{edges incident to } v_i, & \text{if } i = j \\ -1, & \text{if } i \neq j \text{ and } \{v_i, v_j\} \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

SANITY CHECK. Verify (if it's not already clear to you) that $L(G)$ is a symmetric matrix with determinant 0.

DEFINITION 5.2. The **reduced Laplacian** $L_0(G)$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and first column of $L(G)$.

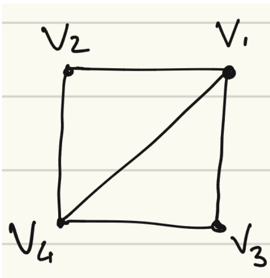

Figure 5:

EXAMPLE 5.3. Let G be the graph in Figure 5. Then

$$L(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Observation. We already know that $\det(L(G)) = 0$, but we note that $\det(L_0(G)) = 8$. This is interesting because if you do a little counting you can verify that G has exactly eight spanning trees.

Theorem 5.4 (Kirchoff's Matrix Tree Thm). For any simple graph G,

$$\det(L_0(G)) = \#\text{ spanning trees of } G$$

Proof. Omitted. □

Remark. The statement of Kirchoff's (yes, the same one of circuit fame) MTT is already absurd on the surface, but perhaps even more absurd is that it doesn't matter how we construct the reduced Laplacian, i.e. if we delete ANY row and ANY column we get the same result.

We now describe a variant of the above (well-known) result to fit our uses.

Definition 5.5. The **polynomial Laplacian** $\mathbb{L}(G)$ of G is an $n \times n$ matrix with entries in $\mathbb{Z}[x_{e_1}, x_{e_2} \dots x_{e_m}]$ given by:

$$\mathbb{L}(G) = \begin{cases} \sum_{e \text{ incident to } i} x_{e_k} & \text{if } i = j \\ -x_{e_k} & \text{if } i \neq j, (i,j) = e_k \\ 0 & \text{otherwise} \end{cases}$$

We define $\mathbb{L}_0(G)$ as the matrix obtained by deleting the first row and column from $\mathbb{L}(G)$. This is known as the **categorification** of the Laplacian.

Observation. If we replace all the variables x_e in $\mathbb{L}(G)$ with the value 1 we recover the original Laplacian $L(G)$.

Theorem 5.6 (Matrix Tree Theorem, polynomial edition). The determinant of $\mathbb{L}_0(G)$ is precisely the spanning tree polynomial Q_G

Definition 5.7. Let H be a graph. Let AH denote a graph obtained from H by adding an **apex** vertex, i.e a vertex with an edge to every vertex of H.

Example 5.8. If $H = K_n$, then $AH = K_{n+1}$

Let H be a graph and let Y_{AH} be the scheme associated to Q_{AH}.

Observation. By the Matrix Tree Theorem (polynomial edition):

$$|Y_{AH}(\mathbb{F}_q)| = \#\{\text{ solutions to } \det(\mathbb{L}_0(AH)) = 0\}$$

Proposition 5.9. $|Y_{AH}(\mathbb{F}_q)| = \#$ symmetric $n \times n$ matrixes A over $\mathbb{F}_q$ with $\det(A) = 0$ such that $A_{ij} = 0$ whenever $i \neq j$ and $(i, j) \notin E(H)$

Proof. Think about it a little. This is purely an exercise in unravelling definitions - spell out explicitly what each of the conditions means algebraically and you should get cryptomorphic sentences. □

Definition 5.10. For any graph H with n vertices, let Z_H be the scheme of $n \times n$ symmetric matrixes of non-zero determinant with zero entries $A_{i,j} = 0$ if $(i, j) \in E(H)$.

The claim then implies that:

$$\left|Y_{AH}(\mathbb{F}_q)\right| = q^{|E(H)|} - |Z_{H^c}|,$$

where H^c is the graph complement of H.

Thus, if Kontsevich's conjecture holds, then all Z_H are polynomially countable

Lecture 6

The Grassmanian

QUESTION. How many (ordered) bases does $\mathbb{F}_q^n$ have?

The answer is as many $n \times n$ invertible matrices there are over $\mathbb{F}_q$. Recall that the columns of a square matrix tell us the image of a given set of basis vectors (which we may assume to be the canonical basis vectors) of that vector space under the linear transformation. And if a matrix is invertible then the images of the canonical basis vectors must themselves constitute a basis. Thus we have an exact bijection between invertible square matrices and ordered bases.

Recall from ?? that we can think of the group (for now we don't care about the group structure specifically) of invertible square matrices as a scheme, with points over any field.

OBSERVATION. We note that:

$$|\mathrm{GL}_n|(q) = (q^n - 1)(q^n - q)\left(q^n - q^2\right)\cdots\left(q^n - q^{n-1}\right),$$

which follows from a standard complementary counting argument: for our first basis element we can choose any nonzero vector (of which there are $q^n - 1$), and then for our second basis element we can pick vector not in the span of the first (note any vector spans q vectors), and so on.

We conclude that GL_n is a polynomially countable scheme.

EXERCISE 6.A. How many *unordered* bases of $\mathbb{F}_q^n$ are there?

DEFINITION 6.1. The given an n-dimensional vector space V, and $k \leq n$, the **Grassmanian** $Gr(k, V)$ is a space that parametrizes all the k-dimensional subspaces of V. We assert that there is a scheme $Gr(k, n)$ defined over $\mathbb{Z}$ such that $\mathrm{Gr}(k, n)(K) = \#\{k\text{-dimensional subpaces of } K^n\}$. (Yes the little k and big K notation here is annoying, apologies for that).

ASIDE FOR EXPERTS. Often the Grassmanian is defined as a functor $\mathrm{Gr}(k, n) : \underline{\mathsf{Sch}} \to \underline{\mathsf{Set}}$ which associates to a scheme S a set of isomorphism classes of quotients

$$\mathcal{O}_S^{\oplus n} \twoheadrightarrow M,$$

where M is a finite, locally free, $n - k$-dimensional $\mathcal{O}_S$ module. Thus instead of parameterizing k-dimensional subspaces it technically paramaterizes $n - k$-dimensional quotients (which is more con-

venient for some technical reasons).
It turns out this functor is *representable* by a scheme, which is denoted the same.

Let us compute the number of $\mathbb{F}_q$-rational points of the Grassmanian scheme. Consider the map:

$$\phi : \{(v_1, \dots, v_k) \text{ lin. indep. ordered list in } \mathbb{F}_q^n\} \to \mathrm{Gr}(k,n)(\mathbb{F}_q)$$
$$(v_1, \dots, v_k) \mapsto \mathrm{span}(v_1, \dots, v_k) \quad (2)$$

Note the cardinality of our domain is $(q^n-1)\cdot(q^n-q)\cdots(q^n-q^{k-1})$. Moreover, for any k-dimensional subspace $U \subseteq \mathbb{F}_q^n$, note that

$$|\phi^{-1}(U)| = \#\text{ bases of } U = |\mathrm{GL}_k|(q).$$

Dividing the cardinality of the domain by the cardinality of each pre-image we get:

$$|\mathrm{Gr}(k,n)|(\mathbb{F}_q) = \frac{(q^n-1)\cdot(q^n-q)\cdots(q^n-q^{k-1})}{|\mathrm{GL}_k(q)|} \quad (3)$$

To write down some neat expressions for this value, given $n \geq 0$, we define the *q-analogues* of the following classical mathematical constructions:

- $(n)_q := 1 + q + \dots + q^{n-1} = \frac{1-q^n}{1-q}$.
- $(n)!_q := (n)_q(n-1)_q...(1)_q$.
- $\binom{n}{k}_q := \frac{(n)!_q}{(k)!_q(n-k)!_q}$.

Remark. The q-analogues are more general combinatorial tools where we replace a standard matheamtical notation with some sort of expression involving q that approaches the original notation as $q \to 1$. For example, you can check that are the notations we introducted q-analogues for so far return the value of the original notation as q gets arbitrarily close to 1.

Proposition 6.2. $\mathrm{Gr}(k,n)(\mathbb{F}_q) = \binom{n}{k}_q$.

Proof. Expand out the expression in 3 and expand out the q-anaologue of the binomial coefficient and see they are the same.eeee

□

Example 6.3. We note that:

$$|\mathrm{Gr}(2,4)|(\mathbb{F}_q) = \binom{4}{2}_q = q^4 + q^3 + 2q^2 + q + 1. \quad (4)$$

QUESTION. What do the coefficients of the point-counting function of the Grassmanian represent, e.g. in Example 6.3?

To answer we recall the following fact from linear algebra: for any k-dimensional subspace $U \subseteq K^n$, there exists a (unique!) ordered basis $u_1, \dots, u_k$ for U such that the matrix

$$\begin{bmatrix} -- & u_1 & -- \\ -- & \vdots & -- \\ -- & u_k & -- \end{bmatrix}$$

is in row-reduced echelon form. Thus we have a bijection between the K-rational points of $\mathrm{Gr}(k, n)$ and k-rank $k \times n$ matrices in row-reduced echelon form.

EXAMPLE 6.4. Let us write down all the possible row-reduced echelon form 2×4 matrices with rank 2.

1. $\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}$
2. $\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$
3. $\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$
4. $\begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
5. $\begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
6. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

where the $*$'s mean we can put any $\mathbb{F}_q$ element in that entry. One can check that the above cover all possibilities. Note we have 1 matrix-type with 4 $*$'s, and so there are q^4 matrices of that type. Similarly we have q^3 matrices of type 2, $2q^2$ matrices of types 3 and 4, q matrices of type 5 and one matrix of type 6. This lines up exactly with our expression in 6.3.

EXERCISE 6.B. Show that:

$$|\mathrm{Gr}(k, n)| = \frac{|GL_n|}{|\mathrm{GL}_k| \cdot |\mathrm{GL}_n| \cdot q^{k(n-k)}}.$$

Deduce that the Grassmanian scheme is polynomially countable.

Lecture 7

Matroid Representation Schemes

Goal: For a matroid M, a scheme of all possible representations of M.
To achieve this, we generalize the notion of an affine scheme.

DEFINITION 7.1. A **quasi-affine scheme** over $\mathbb{Z}$ (of finite type!!) is the data of two finite lists of polynomials in finitely many variables: $f_1..f_s, g_1...g_t \in \mathbb{Z}[x_1...x_n]$. For any field K we define the K-**rational points** of X to be

$$X(K) = \{(a_1...a_n) \in K^n : f_i(a_1...a_m) = 0 \forall i\} \setminus \{(a_1...a_m) \in K^m : g_j(a_1...a_m) = 0 \forall j\}$$

EXAMPLE 7.2. For $m = 2$, with no function fs, but $g_1 = x_1, g_2 = x_2$. Then, $X(\mathbb{R}) = \mathbb{R}^2 \setminus \{0, 0\}$. Without some abstract nonsense, it is a non-trivial fact that this scheme is NOT affine (i.e the punctured plane cannot be written as the vanishing locus of finitely many 2-variable polynomials).

ASIDE FOR EXPERTS. For readers more familiar with algebraic geometry, a quasi-affine scheme is (isomorphic to) a quasicompact, open subscheme of an affine scheme. The way we've characterized it then our f's comprise the data of the affine scheme and the g's comprise the data of the closed subscheme that is the complement of our quasi-affine scheme.
An interesting note: one can show that an equivalent characterisation of a q.a. scheme is one such that the canonical map $X \to \text{Spec}(\Gamma(X, \mathcal{O}_X))$ is a quasicompact topological open immersion.

Given a matroid M on ground set $[n]$ and given $s \in \mathbb{Z}^+$, we claim that there exists a quasi-affine scheme $\text{Rep}(s, M)$ such that for any field K, we have:

$$\text{Rep}(s, M)(K) \cong \{f : [n] \to K^s : \forall T \subseteq [n], \text{rank}(f(T)) = \rho_M(T)\}$$

where $\rho_M(T)$ is the rank function of M.

In other words, the points of $\text{Rep}(s, M)(K)$ are in bijection with all the possible ways to represent M in K^s. Consider an $n \times s$ matrix A of ns variables:

$$A = \begin{bmatrix} --- & v_1 & --- \\ --- & v_2 & --- \\ & \vdots & \\ --- & v_n & --- \end{bmatrix}$$

For each $T \subseteq [n]$, let A_T be the submatrix of rows indexed by elements of T. Let $r = \rho_M(T)$. Turn the determinants of the $(r + 1) \times (r + 1)$ minors into the fs. For gs, use the polynomials of the form

$\prod_T (\rho_M(T) \times \rho_M(T)$ minor of $A_T)$.

This gives us a scheme for $\text{Rep}(s, M)$ because the f polynomials being zero tells us that any set of $> \rho_M(T)$ vectors is linearly independent. The condition of the gs *not* all being zero is equivalent to at least one set of r vectors such that they are linearly independent.

Example 7.3 (Representations of the Fano Matroid). Take the Fano Matroid F_7. We know that this is representible over K if and only if $char(K) = 2$. Thus, $|\text{Rep}(3, F_7)|$ is not a polynomial. For q a power of 2 let us count the total number of representations, i.e compute $|Rep(3, F_7)|(q)$.

Using the notation as in Figure 4, we have $q^3 - 1$ options for a. Recall that we are scaling all the vectors of the matroid's representations to a common plane, so in order to visualize b as a distinct point from a we want it to not lie in tis span. Since the span of a has q elements, we then have $(q^3 - q)$ options for b, and consequently $(q^3 - q^2)$ options for c.
By similar logic we deduce d should not be in the span of any combination of a, b or c, so using standard Inclusion-Exclusion couting arguments we get $(q^3 - 3q^2 + 2q - 1)$ options for d.
We know e has to lie in the span (a, b) and the span of (c, d), which is essentially two planes intersecting in a line. Thus, we have $q - 1$ non-zero options for e, and likewise for f and g.
As we are working on characteristic 2, g being in the span of (a, d) and (b, c) automatically puts it in a line of points within the span (e, f), so the final condition is automatically satisfied. Thus we conclude:

$$|Rep(3, F_7)|(q) = (q^3 - 1)(q^3 - q)(q^3 - q^2)(q^3 - 3q^2 + 3q - 1)(q - 1)^3$$

Lecture 8

Projective Geometry

One can think of affine n-space as our standard, intuitive concept of n-dimensional space - all possible ordered lists of length n with values in K. The more one does algebraic geometry, however, the more one realizes that this is, in many ways, the wrong space to be using. A simple example is that in standard affine space, we have an "almost-symmetry" of points and lines. Every pair of points defines exactly one line, and MOST pairs of lines "define" (by intersecting at) a point. There is however the pesky issue of parallel lines. To fix this (and several other not-so-nice issues that arise), we work over *projective space.*

Definition 8.1. We denote by $\mathbb{P}^n_K$ the **projective n-space** over K to be the space of 1-dimensional subspaces of K^{n+1}.

Observation. $\mathbb{P}^n_K = \text{Gr}(1, n+1)(K)$

Convention: If $x_0, x_1 ... x_n \in K$ not all 0, write $[x_0 : x_1 : .. : x_n] \in \mathbb{P}^n_K$ for the line spanned by $(x_0, x_1 ... x_n) \in K^{n+1}$. By definition, such a representative of a point in projective space is unique up to scaling all the coordinates by some (nonzero!) element of the field. Thus we may also define $\mathbb{P}^n_K$ as:

$$\left\{\vec{x} = (x_0, \dots, x_n) \in K^{n+1}\right\} \big/ ((x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n),\ \forall \lambda \in K^\times)$$

Exercise 8.A. Show that $|\text{PGL}_n|(q) = \frac{|\text{GL}_n|(q)}{q-1}$.

$\mathbb{P}^2_k$ is called the projective plane over K. Given distinct $[a : b : c], [d : e : f] \in \mathbb{P}^2_k$, there is a unique *projective line* containing them.

Definition 8.2. A **projective line** in $\mathbb{P}^2_k$ is the set of all lines through 0 in a 2-dim subspace of K^3

Example 8.3. Let us calculate the coordinates of $\overline{v_1 v_2} \cap \overline{v_3 v_4} \in \mathbb{P}^2_k$, where:

- $v_1 = [1 : 0 : 0]$.
- $v_2 = [0 : 1 : 0]$,
- $v_3 = [0 : 0 : 1]$.
- $v_4 = [1 : 1 : 1]$.

Note that x, y, z must satisfy:

$$\det \begin{bmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \det \begin{bmatrix} x & y & z \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 0,$$

which is a determinant-esque way of saying (x, y, z) must be in the subspaces of K^3 spanned by the pairs v_1, v_2 and v_3, v_4 respectively. With a bit of calculation one can verify the above conditions require $z = 0$ and $x = y$. Since projective coordinates are invariant under scalars, we indeed verify that the lines meet at the unique point $[1 : 1 : 0]$.

Remark. A novice yet careful reader will have noticed that when talking about projective space we oscillate back and forth between the different (yet equivalent!) descriptions of $\mathbb{P}^n_K$ a lot. This is standard and useful practice.

Given a set of starting points in the projective plane $\mathbb{P}^2_K$, we can draw lines between any two points, and we can "create" new points by taking the intersection points of two lines. To this end, we say a point $[x : y : z] \in \mathbb{P}^2_K$ is able to be **marked** if it arises as the intersection point of two lines arising from the aforemention process. E.g in 8.3 given v_1, v_2, v_3, v_4 as starting points we were able to mark $[1 : 1 : 0]$, which we can visualize pictorially in Figure 6.

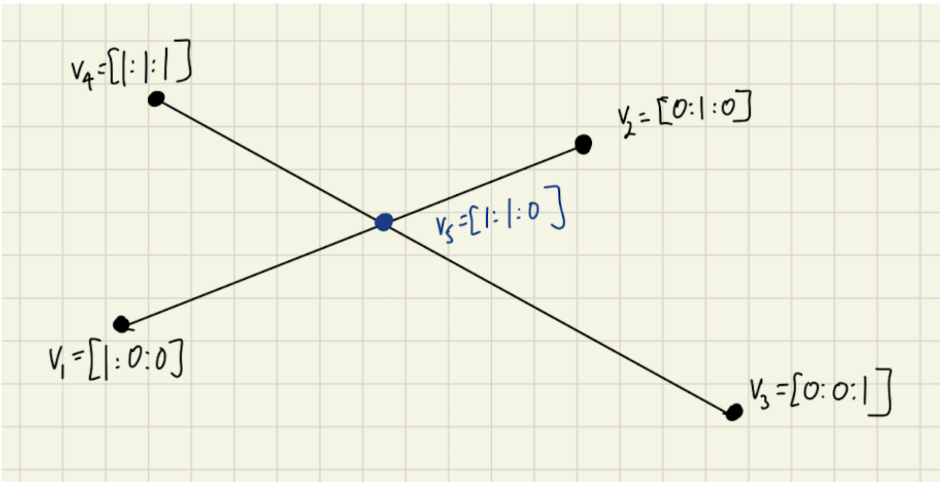

Figure 6: Marking the point v_5 with our 4 starting points.

Let us start with the points v_1, v_2, v_3, v_4 as in 8.3 and $s := [1 : 0 : x]$ and $t := [1 : 0 : y]$. Our goal will be to mark the point $[1 : 0 : x + y]$.
Given $v_1 = [1 : 0 : 0]$ and $v_2 = [0 : 0 : 1]$, the points $[1 : 0 : \lambda]$ is collinear with v_1, v_2 for any $\lambda \in K$, since, as a 3-vector, we can write it as $v_1 + \lambda v_2$. Now, we mark new points $[0 : 1 : -x]$ and $[1 : 1 : y]$ drawing new lines in our previous construction, as shown in Figure 7.

We can verify that the points v_6 and v_7 really do lie on their respective lines, as they should be able to be expressed as a linear combination of other points on their lines. In particular $v_6 = v_5 - s = v_2 - xv_3$ and $v_7 = v_2 + t = v_4 + (y - 1)v_3$.

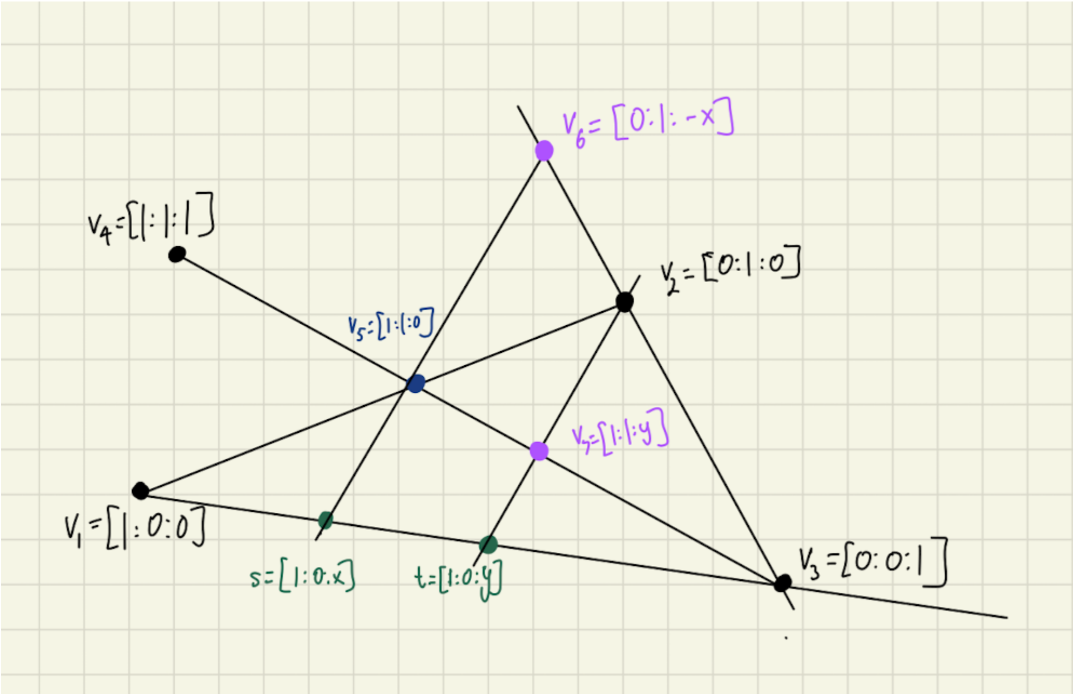

Figure 7: Marking points v_6 and v_7.

EXERCISE 8.B. By drawing one more line in Figure 7, mark the point $[1 : 0 : x + y]$.

Lecture 9

Projective Matroid Representations

A linear map T preserves the rank of any set of vectors. Moreover, if T is invertible, nothing except the 0 point gets sent to 0, and so it induces a map in projective space. If it was not invertible, it couldn't since 0 is not a point in projective space.

Thus, all the standard matrix groups and schemes.

PROPOSITION 9.1. Take $v_1, v_2, v_3, v_4 \in K^3$ such that any 3 are linearly independent. Then, there is a unique (up to non-zero scaling) invertible linear map $T : K^3 \to K^3$ such that:

- $T(\langle v_1 \rangle) = [1 : 0 : 0]$
- $T(\langle v_2 \rangle) = [0 : 1 : 0]$
- $T(\langle v_3 \rangle) = [0 : 0 : 1]$
- $T(\langle v_4 \rangle) = [1 : 1 : 1]$,

where $\langle v_i \rangle$ is the span of v_i.

Proof. Think about it. □

DEFINITION 9.2. Let M be a matroid on the set $[n]$. Points $p_1, ...p_n \in \mathbb{P}^m_k$ are a **projective representation** of M if $\operatorname{rank}(p_{i_1}, ..p_{i_s}) = \rho_M(\{i_1, ...i_s\})$ for every indexing set $\{i_1, \dots, i_s\} \subseteq [n]$.

PROPOSITION 9.3. Given a matroid M on $[n]$:

$$\# \left\{ \begin{array}{l} \text{projective reps } p_1, ...p_n \in \mathbb{P}^2_K \text{ of } M \text{ such that} \\ p_1 = [1 : 0 : 0], p_2 = [0 : 1 : 0], p_3 = [0 : 0 : 1], p_4 = [1 : 1 : 1] \end{array} \right\} \cdot |\mathrm{PGL}_2(K)|$$
$$= \# \left\{ \begin{array}{l} \text{projective reps } p_1, ...p_n \in \mathbb{P}^2_K \text{ of } M \text{ such that} \\ p_1, p_2, p_3, p_4 \text{ are in general position (i.e no 3 on a line)} \end{array} \right\}$$

Proof. Follows from property of linear maps acting on points in projective spaces, corresponding to lines in K^{n+1}. □

OBSERVATION. If M has no **loops** (i.e singletons with rank 0), then:

$$\#\{\text{ projective reps of } M \text{ in } \mathbb{P}^n_k\} \times (q-1)^n = \{\#\text{ reps of } M \text{ in } K^{n+1}\} \tag{5}$$

Lecture 10

Mnëv Universality

Definition 10.1. A **partial rank function** on $[N] = \{1, 2, \ldots, N\}$ is a set of subsets $S \subseteq \wp([N])$ and a function $f : S \to \mathbb{Z}_{\geq 0}$.

We introduce this notion as it is sometimes eas

Theorem 10.2 (Mnev Universality). Let X be an affine scheme over $\mathbb{Z}$ corresponding to $f_1..f_s \in \mathbb{Z}[x_1...x_n]$. Then, there is some $N \in \mathbb{N}$ and a partial rank function ρ on subsets of $[N]$ such that:

$$\sigma \cdot |X| = |\operatorname{Rep}(3, \rho)|,$$

where σ is some polynomial of q.

Proof. We draw a configuration of points and lines in $\mathbb{P}^2$ starting with $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]$. □

Corollary 10.3. For any affine scheme X:

$$\sigma \cdot |X| = \sum_{M \in \zeta} |\operatorname{Rep}(3, M)|,$$

For ζ some finite set of matroids (specifically the matroids that are extension of the given partial rank function to a full rank function) and σ some polynomial.

We now define some objects that seem completely arbitrary and convoluted, but whose properties will be vital in our proof (or disproof?) of Kontsevich's conjecture.

(1) Let Mat (short for **matroids**) be the set of all functions $f : Q \to \mathbb{Z}$ of the form:

$$\alpha_1 |\operatorname{Rep}(s_1, M_1)| + \cdots + \alpha_t |\operatorname{Rep}(s_t, M_t)|,$$

where $\alpha_i = \frac{\beta_i(q)}{\gamma_i(q)}$, where $\beta_i, \gamma_i \in \mathbb{Z}[q]$ such that γ_i has no zeroes in Q.

(2) Let Mot (short for **motives**) be the set of all funcitions of the form:

$$\alpha_1 |X_1| + \cdots \alpha_t |X_t|,$$

where $X_1 ... X_t$ are any quasi-affine schems, and α_i are as before.

(3) Let Graph (you can guess what this is short for) be the set of all funtions $f : Q \to \mathbb{Z}$ of the form:

$$\alpha_1 |X_{G_1}| + \cdots + \alpha_t |X_{G_t}|,$$

where the α_i are as before and each X_{G_i} is the scheme associated to the spanning tree polynomial Q_{G_i} of some graphs G_i.

OBSERVATION. Note Mat $\subseteq$ Mot and Graph $\subseteq$ Mot by definition.

COROLLARY 10.4. By Mnëv Universality,

$$\mathsf{Mat} = \mathsf{Mot}$$

One more construction we will need is the notion of an *incidence scheme*. First we recall a **bilinear form** on K^n is a function $B : K^n \times K^n \to K$ that is linear in each coordinate. The canonical example of one is if A is an $n \times n$ matrix over K then the map

$$(v, w) \mapsto v^T Aw \tag{6}$$

is a bilinear form on K^n. Recall the scheme Sym_r^n defined in 1.6.

SANITY CHECK. In the above example, what happens when $A = I$? Is this bilinear form something you are familiar with?

DEFINITION 10.5. Fix a graph G, and $n \in \mathbb{Z}_{>0}$ and $k, r \in \mathbb{Z}_{\geq 0}$. We define the **incidence scheme** $\mathrm{Sym}_r^n f_k^G$ be the scheme of pairs (Q, f) such that:

- $Q \in \mathrm{Sym}_r^n$,
- f is a function $V(G) \to K^n$ such that $f(V(G))$ has rank k.
- $Q(f(v), f(w)) = 0$ whenever $\{v, w\} \in E(G)$,

where in the third bullet points Q is seen as a bilinear form in the same way as in Equation 6.

SANITY CHECK. Verify that $\mathrm{Sym}_r^n f_0^G \cong \mathrm{Sym}_r^n$ (by $\cong$ here we mean that these two schemes have the same K-rational points for any K).

We now have all the pieces set. In the next lecture we will see how we put them all together.

Lecture 11

Conclusion

Let us review the concepts and objects we've constructed so far. We let G be a connected graph, and let X_G be the finite-type affine $\mathbb{Z}$-scheme associated with Q_G, the spanning tree polynomial defined in 2.7.

Given a set $\mathscr{F}$ of functions $Q \to \mathbb{Z}$ we introduce the notation

$$\langle \mathscr{F} \rangle = \{\sigma_1 F_1 + \cdots + \sigma_m F_m \ : \ F_i \in \mathscr{F}, \sigma_i \text{ rational functions in } \mathbb{Z} \text{ with no denominators in } Q\},$$

i.e all finite linear combinations of these functions. Then, we have:

- $\mathsf{Graph} = \langle\, |X_G| \ : \ G \text{ is a graph}\rangle$
- $\mathsf{Mat} = \langle\, |\mathrm{Rep}(s, M)| \ : \ s \in \mathbb{Z}_{\geq 0},\ M \text{ a matroid}\rangle$
- $\mathsf{Mot} = \langle\, |X| \ : \ X \text{ any quasi-affine scheme}\rangle$

By Mnëv Universality (Theorem 10.2), $\mathsf{Mat} = \mathsf{Mot}$.

Theorem 11.1 (Belkale-Brosnan). With the notation as above,

$$\mathsf{Graph} = \mathsf{Mat}$$

Corollary 11.2. Kontsevich's conjecture is false.

Proof. If all the point-counting functions $|X_G|$ were polynomials, then any $F \in \mathsf{Graph}$ would be a rational function over $\mathbb{Z}$. Assuming 11.1, this means any $F \in \mathsf{Mot}$ is a rational function over $\mathbb{Z}$, i.e the point-counting function of any quasi-affine $\mathbb{Z}$-scheme is a rational function. However, we've seen in Examples such as 1.5 and **??** shows this is not the case, and so Kontsevich's conjecture is false. □

Remark. In fact, 11.1 is a bit more insulting to Kontsevich. Forget about being polynomially countable (which is actually quite a tight restriction), Belkale and Brosnan tell us that the types of schemes (and their respective point-counting functions) generated by connected graphs are *as general as possible*, and thus they follow the so-called Murphy's Law.

Sketch of proof of 11.1. Let G have n vertices, and define Z_G the same as in 5.10. Stanley (of combinatorial fame) showed that $\langle\, |Z_G| \ : \ G \text{ is a graph}\rangle = \mathsf{Graph}$.
Belkale and Brosnan's key result was showing:

$$\left\langle\, \left|\mathrm{Sym}^s_r\, f^G_k\right| \ : \ G \text{ any graph}, s, r, k \in \mathbb{Z}_{\geq 0} \right\rangle = \mathsf{Graph}\,.$$

From this it was relatively clear to show the LHS above is equal to Mat, which completes the theorem. □

Author note: at some point I may flesh out the proof sketch above in a bit more detail

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