Proofwriting Minicourse Notes

Aaryan, Ellie, Thea

PROMYS 2022

(Advice for material by professors at Rose-Hulman Institute of Technology, in particular, Drs. Joshua Holden, Ralph Grimaldi, William Green, Tracy Weyand, Manda Riehl, Slyvia Carlisle, Leanne Holder, Timothy All, and David Finn. Originally written for PROMYS 2019, edited by Lee Trent for PROMYS 2019, edited and TEX'ed for PROMYS 2022)

1 What is a Proof

In a word, a proof is meant to be **convincing**. When writing a paper, you have to convince an advanced mathematician that your result is correct. When writing PROMYS Psets, you have to convince your counselor that you understand the material.

Proofwriting can be difficult! You've done math before, but you may not have written proofs. Proofs are how we show that we know something for certain, not just that something is true about every example we've checked. Proof are rigorous - that means they are thorough, exhaustive, and accurate. When you write a proof, every step must be fully justified, and your starting information must be axioms, definitions, and/or theorem that are agreed upon between you and the person reading your proof. IF you've never seen proofs before, your first few (or many) will have holes in them - steps that you have not fully justified within the proof. When your counselor points these out, don't feel bad about it. You guys are hear to learn and we fully expect this to be how your first proofs turn out. It's how ours did too! You're here not just to learn mathematical facts, but also (and more importantly) mathematical, rigorous, critical ways of thinking. Be bold and thorough in your arguments and conjectures, and we'll be there to help you refine them.

2 Proofwriting Basics

2.1 Logical Quantifiers

In mathematics, we have shorthand symbols for certain phrases often used in statements of conjectures and proofs. Here are some of the most common, written in the language of **first-order set theory**:

- $\implies$ - This symbol means **implies**. $A \implies B$ means that if A is true, then B is necessarily also true (but not necessarily the converse!) An example is $x \textit{ is even} \implies 3x \textit{ is even}$

- $\iff$ - This double-sided arrow means **if and only if** (sometimes abbreviated to **iff**), and is used for two way implications. In other words, $A \iff B$ means $A \implies B$ and $B \implies A$. An example is x is even $\iff x^2$ is even

- $\forall$ - This symbol means **for all.** An example is $\forall \epsilon < 0$, $\sqrt{\epsilon}$ is not a real number

- $\exists$ - This symbol means **there exists**. An example is $\forall \epsilon > 0, \exists \epsilon_1$ such that $0 < \epsilon_1 < \epsilon$

- $\wedge$ and $\vee$ - These are the symbols for **and** and **or**, respectively. We could write N prime $\implies$ (N odd) $\vee (N = 2)$

- $\neg$ - This is the symbol for **negation**, and $\neg(A)$ is read as **"not** A**"**. For example, we can write $(N > 3) \wedge (N$ prime$) \implies \neg(N$ even$)$

In practice, you won't actually use these symbols all the time, as it is often easier and clearer to simply use the words, but they do pop up, and they allow for more concise writings of complex statements sometimes.

2.2 DeMorgan's Laws

Let A and B be mathematical statements. Then, DeMorgan's Laws give us the simple rules of logic:

$$\neg(A \vee B) \iff (\neg A) \wedge (\neg B) \tag{1}$$

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B) \tag{2}$$

2.3 Proof Terms

Hierarchy

There is a certain hierarchy of mathematical results, and it is important to be aware of the different meanings and significance of these terms.

- **Theorem** - A big, standalone desirable result that is proven to be true. For example, Pythagoras' theorem.

- **Proposition** - This is a statement that isn't *as* big or important as a theorem, but is enough to stand on its own.

- **Lemma** - An intermediary step used in the proof of a larger theorem. In the proof of Pythagoras, you may use the lemma that the area of a right triangle with legs of length a and b is $\frac{ab}{2}$

- **Corollary** - A result that is a (somewhat trivial) consequence of a big theorem or result. A corollary of Pythagoras could be the fact that in a right angle triangle, the hypotenuse is longer than either of the legs but shorter than their sum.

- **Axiom -** The underpinnings of mathematical reasoning. These are statements taken to be true at face value, without any 'proof'. For example: *If two sets A and B have the same elements, then they are the same set.* There is nothing we can do to *prove* this statement, it is just declared inherently true. Working with different sets of axioms can lead to different mathematics emerging!

You'll notice some of the lines between these terms seem to be a bit blurry and vague, and that's true! Often it's up to the author to decide what is and isn't important enough to be a theorem or proposition.

3 Some Methods of Proofs

3.1 Direct Proof

Given a conditional statement, or in symbols a statement where $p \implies q$, we assume that p is true and from that assumption prove that q is also true.

3.2 Proof by Contradiction

Again, given a conditional statement, $p \implies q$, we assume p and $\neg q$, and follow through the logical conclusions until we arrive at a statement which we know (from the original hypotheses, $p, \neg q$, or from previously established results) to be false. Then, we have contradicted that q can be false while p is true, and therefore, $p \implies q$. (This is because $\neg(p \wedge \neg q) \iff (p \implies q)$, which you can check with a truth table.)

3.3 Proof by Contrapositive

Using a truth table, you can show that $(p \implies q) \iff (\neg q \implies \neg p)$. So if we want to show that $p \implies q$, we could instead show $\neg q \implies \neg p$, which is sometimes simpler than showing directly that $p \implies q$. From here, one can use the method of direct proof (or contradiction if you want to make things complicated.

4 Proofwriting Tips

(Many of these are from lists of tips given to real college students by their real professors.)

- Be clear about what is and isn't part of your proof. Write down the statement you intend to prove, including the necessary assumptions. Then, begin your proof with “Proof.” and end it with "$\square$" "QED," or something similar.

- Use symbols correctly. Mathematical symbols have an English translation- read your work out loud to be sure it makes sense. For example, $\exists x \in \mathbb{R}$ means“there exists an x in the set of real numbers.” On the other hand, $\exists \mathbb{R} x$ doesn't make sense because $\mathbb{R}$ is a noun that means “the set of real numbers,” not an adjective that means “real.”

- Be precise. Did you mean $\mathbb{Z}$ or $\mathbb{N}$? Or perhaps $\mathbb{N} \cup 0$? [1]
- Use complete thoughts and reasoning. Every statement in your proof should have justification. Indicate whether it comes from a definition, a theorem you have already proved, one of your initial assumptions, an axiom, et cetera. State your assumptions early.
- Use lots of space. Use lots of paper, or digital pages, to show your process. Be detailed.
- Yes, even more detailed than that. Err on the side of overjustifying, and pull back from that after your counselor lets you know you can.
- For the sake of your counselors, write legibly, spread out your work, and make it clear where problems and proofs begin and end. Imagine your counselor wanted to make a comment on every line of your proof- there should be room to do so in the margins or between lines. Sometimes in your life your proofs will just be for you, but in PROMYS, on your number theory assignments, your proofs are a collaborative effort between you and your counselor, where your input is a proof, their input is feedback, and the goal of the project is your growth. Make space for your partner, your counselor, on the page.
- Balance use of mathematical notation and English words. Is the next thing you want to present more clear in English or mathematical notation? Perhaps it's best to introduce it with English and state it with mathematical notation. This is not a science. Practice, and your counselor will give you feedback.
- Notation can be very information dense. Inequalities, Equations, and other statements in mathematical notation can be spaced out (placed on their own lines, for example) to help both you and the reader follow the flow of information more easily.
- When citing a definition or axiom, make sure you're actually using that definition or axiom and not something related or derived from it. Similarly for theorems, with the additional requirement that when using a theorem you need to be sure the conditions for application of the theorem have been met.A theorem may be "if x then y" and if you want to use it, you need to explain why x has been met so you know that y is true
- When in doubt, explain your work. Unless the statement is a theorem, make sure you explain how you have gotten to the next step of your logic
- (Shoutout to Henry Cohn for this great advice) A key principle of proof writing is that it is a method meant for communication between humans, not some kind of derivation meant to be checked by a machine. We want to convey understanding with a proof, not just a "correct" result. With that said, a proof should follow a narrative flow instead of being isolated statements. This may be hard to do at first, but one of the main parts of being a mathematician is conveying your understanding to others in a clear fashion.

[1] While the author does not agree with this and there is constant debate in the mathematical community, in PROMYS $0 \notin \mathbb{N}$

5 What is a PODASIP?

PODASIP stands for "Prove or Disprove and Salvage if Possible"

When you encounter "PODASIP: statement," first, determine whether "statement" is true or false. If it is true, prove it! (Remember to be very clear about what you are proving and to write a very clear proof!) Then, if you'd like, make a conjecture, perhaps to generalize it. You can treat your new conjecture as its own PODASIP!
If "statement" is false, disprove it! (Often by providing a counterexample, stating that it is a counterexample, and explaining clearly why it is a counterexample.) Then, salvage if possible! This part is easy to forget, but it's an important part of the mathematical process. Tweak the original statement to create a new statement - your own new fun conjecture - that you think may be true, and try to prove it if you'd like. Again, you've got a new PODASIP!
Examples:

PODASIP: Only solid black cats are cute
(A) SOLUTION: False.
Counterexample: Lee's cat, Lily.
Lily is a black, brown, and white tabby, not solid black. Due to her having pretty eyes and well-proportioned whiskers, Lily is cute. Since she is cute but not solid black, she is a counterexample to the claim.
Conjecture: Only cats with some black on them are cute.

PODASIP: Number theory is cool.
(A) SOLUTION: True.
Proof. Note that there are over 100 young people studying number theory at PROMYS this summer. Theorem 3.14 states that no more than 17 people would study something uncool in the summer. Since $100 > 17$, number theory is cool.

6 Exercises

(These are intended to be things you know, for the sake of practicing writing good proofs.)

- Prove that if A, B are two sets, then $A' \cap B' = (A \cup B)'$, where A' means "everything that is not in set A". This is the set theoretic version of DeMorgan's Laws
- Prove that opposite sides of a parallelogram have equal length.
- Prove that if two sides of a triangle have equal length, then the angles opposite them have equal measure.
- Prove (using a contrapositive!) that if r is irrational, then $\sqrt{r}$ is irrational
- Prove (using contradiction) that $\sqrt{2}$ is irrational.