

Pontryagin Duality

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Basic Topological Notions

DEFINITION 0.1. Let X, Y be topological spaces. Then the **product topology** on the Cartesian product $X \times Y$ is given by setting all the open sets to be those of the form

$$U \times V \subseteq X \times Y,$$

such that U is open in X and V is open in Y . Moreover if we have an equivalence relation $\sim$ on X and $[x]$ denotes the equivalence class of $x \in X$, recall we have the canonical quotient map

$$\begin{aligned} q : X &\rightarrow X / \sim \\ x &\mapsto [x]. \end{aligned}$$

The **quotient topology** on $X / \sim$ is given by setting all the open sets to be U such that $q^{-1}(U)$ is open in X .

The product and quotient topologies satisfy the following "universal properties".

EXERCISE 0.A. Prove the following.

(a) For a product of topological spaces $X_1 \times X_2$, the projection maps

$$\begin{aligned} \phi_i : X_1 \times X_2 &\rightarrow X_i \\ (x_1, x_2) &\mapsto x_i \end{aligned}$$

are continuous.

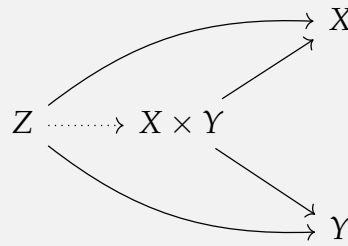
(b) For a quotiented topological space $X / \sim$, the quotient map

$$\begin{aligned} q : X &\rightarrow X / \sim \\ x &\mapsto [x]. \end{aligned}$$

is continuous.

In fact, something stronger than the above exercise holds. We won't prove this next result, but it will come in useful.

PROPOSITION 0.2. 1. For X, Y and Z topological spaces and $X \times Y$ given the product topology, if we have continuous maps $Z \rightarrow X$ and $Z \rightarrow Y$, there exists a unique continuous map $Z \rightarrow X \times Y$ that commutes with the projection maps, i.e the diagram



commutes.

- For X, Z topological spaces and $\sim$ an equivalence relation on X , and a continuous map $f : X \rightarrow Z$ such that $f(x) = f(y)$ if and only if $x \sim y$, we have a unique bijection $\tilde{f} : X / \sim \xrightarrow{\sim} Z$ such that $\tilde{f} \circ q = f$.

DEFINITION 0.3 (Topological properties). A topological space X is **Hausdorff** if any two points can be separated by disjoint open sets, i.e

$$\forall x, y \in X, \exists \text{ open } U_x \ni x, U_y \ni y \text{ s.t. } U_y \cap U_x = \emptyset.$$

We say a Hausdorff space X is **locally compact** if every $x \in X$ has an open neighborhood whose closure is compact. This closure is called a **compact neighborhood**.

EXAMPLE 0.4. The rationals $\mathbb{Q}$, with its standard topology, is not locally compact. This is because any open neighborhood $(-a, b)$ of the identity 0 contains some irrational r . Then since $[-a, r)$ and $(r, b]$ is not compact in $\mathbb{R}$ there exists an open cover of both of them that have no finite subcover, and taking their union gives us a cover of $\overline{(-a, b)}$ with no finite subcover.

We can also discuss the notion of local compactness for non-Hausdorff spaces, but we will practically never use such spaces. An important result about Hausdorffness is the following:

PROPOSITION 0.5. A topological space X is Hausdorff if and only if the diagonal

$$\Delta := \{(x, x) : x \in X\} \subseteq X \times X$$

is a closed subset of $X \times X$ (given the product topology).

Proof. For one direction, if you are Hausdorff, then by definition for all $x \neq y \in X$, we have open sets $U_x \ni x, U_y \ni y$ such that $U_x \cap U_y = \emptyset$. Then consider the neighborhood $U_x \times U_y \ni (x, y)$. By assumption that U_x and U_y are disjoint, we deduce there $U_x \times U_y \cap \Delta = \emptyset$. Thus the complement of Δ in $X \times X$ is open, and therefore Δ is closed.

The same argument in reverse proves the other direction as well. □

This is the standard way we will show a topological group is Hausdorff.

DEFINITION 0.6. Given a topological space X with topology τ_X , a family of open sets $\mathcal{B}$ is a **base** (or sometimes also called a **basis**) for the topology if every open set in X can be expressed as the union of sets in $\mathcal{B}$.

The sets in $\mathcal{B}$ are sometimes referred to as **basic open sets**.

Another useful notion will be all the various equivalent ways to verify something is continuous. In different scenarios, one might be easier to verify than another.

THEOREM 0.7 (Equivalent ways to define Continuity!). For a function $f : X \rightarrow Y$ between topological spaces, the following are equivalent:

- (1) f is continuous (i.e. the preimage of any open set is open)
- (2) Given a base $\mathcal{B}_Y$ of Y , the pre-image of any $B \in \mathcal{B}_Y$ is open.
- (3) For any subset $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.
- (4) For every closed $C \subseteq Y$, its preimage $f^{-1}(C)$ is closed in X .
- (5) For every $x \in X$, and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. Unnecessary (we will take this result on faith here), but feel free to do it as an exercise if you wish :)

□

Section 1

Topological Groups

DEFINITION 1.1. A **topological group** G is a group with a topology on it so that the multiplication map $(m : G \times G \rightarrow G)^a$ and inversion map $(i : G \rightarrow G)$ are both continuous.

^ahere $G \times G$ is given the product topology.

EXAMPLE 1.2. Take $\mathbb{R}$ with the usual Euclidean topology, and its additive group structure. You have seen in Pset 2 that the addition map $+$: $\mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, and moreover the inversion map $i : \mathbb{R} \rightarrow \mathbb{R}$ sending $x \mapsto -x$ is also continuous, since it is multiplication by a constant. Thus $\mathbb{R}$ is a topological group.

EXERCISE 1.A. Show any group G with the discrete topology is a topological group.

EXAMPLE 1.3. Take the circle

$$S^1 := \{z \in \mathbb{C} : |z| = 1\} \subseteq \mathbb{C}^\times,$$

with the subspace topology from $\mathbb{C} \cong \mathbb{R}^2$. The group operation and inversion are standard complex multiplication and inversion (hence continuous).

EXERCISE 1.B. Show a group G with a topology on it is a topological group (i.e the multiplication and inversion maps are continuous) if and only if the map

$$\begin{aligned} \kappa : G \times G &\rightarrow G \\ (g, h) &\mapsto gh^{-1} \end{aligned}$$

is continuous.

PROPOSITION 1.4. A set $U \subseteq G$ of a topological group is open if and only if $U = gV$ for some $g \in G$ and V an open neighborhood of the identity.

Proof. Exercise :) □

We will now discuss how to form new topological groups from ones we already know.

EXERCISE 1.C. Let H be a subgroup of a topological group G , given the subspace topology.

- (a) Show H is a topological group (Hint: consider $H \times H \subseteq G \times G$, and use Exercise 1.B).
- (b) Show that the closure $\overline{H}$ is also a subgroup (Hint: recall the subgroup criterion. It suffices to show the image of $\kappa(\overline{H} \times \overline{H})$ defined in 1.B is contained in $\overline{H}$. Use Formulation (3) of 0.7.).

EXAMPLE 1.5. If we take $\mathbb{Q} \subseteq \mathbb{R}$ with the subspace topology, this is the standard topology on $\mathbb{Q}$, and thus $\mathbb{Q}$ with its standard topology is a topological group.

Moreover if we take $\mathbb{Z} \subseteq \mathbb{R}$, note every integer has an open neighborhood that does not intersect any other integer, thus the subspace topology on $\mathbb{Z}$ is the discrete one.

We now use these facts to show that the product and quotient of topological groups are, in fact, topological groups (as we would expect).

EXERCISE 1.D. (a) Given groups G, G' , show that the direct product of groups $G \times G'$, given the product topology, is a topological group.

- (b) Given a normal subgroup $H \trianglelefteq G$, show the quotient group G/H with the quotient topology is a topological group. Use part (b) of 0.A.

EXERCISE 1.E. Show that a homomorphism of topological groups $f : G \rightarrow H$ is continuous everywhere if and only if it is continuous at some arbitrary point $g \in G$ (i.e the preimage of every open neighborhood of $f(a)$ is open in G). [Hint: use the fact that multiplication is continuous!]

KEY IDEA. Since group multiplication is continuous and transitive, if we want some property of the topology or a continuous map to hold everywhere on a topological group, it (generally) suffices to show that property holds locally around some arbitrary point.

EXERCISE 1.F. Let $\mathbb{Q}$ and $\mathbb{R}$ have their standard topologies. Show that any continuous group homomorphism $f : \mathbb{Q} \rightarrow \mathbb{R}$ is uniquely determined by $f(1)$. Deduce (using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$) that any continuous group homomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$t \mapsto rt,$$

for $r \in \mathbb{R}$.

Using the above Key Idea, with Proposition 1.4 will be very useful in the following exercise.

EXERCISE 1.G. Let H be a subgroup of a topological group G . By 1.C, we know this is also a topological group.

- (a) Show that H is open if and only if it contains a nonempty open set U (hint: consider showing $UH = H$).
- (b) Show that if H is open, it is also closed (hint: consider cosets of H , and apply 1.4).

The above shows a strange property of topological groups: any open subgroup is actually clopen!

When learning measure theory, you learned that locally compact Hausdorff spaces are very useful for constructing good measures. We will want these properties on our topological groups as well, so we review necessary and sufficient conditions for these nice topological properties to hold.

EXERCISE 1.H. Show a topological group G is Hausdorff if and only if the singleton set $\{e\}$ is closed (use Prop 0.5).

EXERCISE 1.I. Show that a Hausdorff group G is locally compact if and only if the identity $e \in G$ has a compact neighborhood (use the Key Idea discussed after 1.E).

DEFINITION 1.6. If a topological group G is Hausdorff, locally compact and abelian, we call it an **LCA group**.

REMARK. Note that if we say a group is locally compact in general, we will always implicitly assume it is also Hausdorff. Although one doesn't imply the other, we will never care about local compactness in spaces where it isn't also Hausdorff.

Section 2

The Dual Group

DEFINITION 2.1. Let S^1 be given its standard topological group structure, as in Example 1.3. For G a topological group, we define its **Pontryagin Dual** $\widehat{G}$ as the group of continuous homomorphisms to S^1 , i.e.

$$\widehat{G} := \left\{ \chi : G \rightarrow S^1 \text{ cont. homomorphism} \right\}.$$

Elements of $\widehat{G}$ are called **characters**.

EXERCISE 2.A. Define a group operation on $\widehat{G}$ by

$$\chi_1 \cdot \chi_2 = (g \mapsto \chi_1(g)\chi_2(g)).$$

Show that this makes $\widehat{G}$ into a group.

EXAMPLE 2.2 ($\mathbb{R}$ is self-dual). We will use here the following fact, which you might see in some form in your topology class but we won't prove here: if $\rho : \mathbb{R} \rightarrow S^1$ denotes the winding projection map $t \mapsto e^{2\pi i t}$, then any map continuous map $f : X \rightarrow S^1$ can be lifted to a continuous map $\tilde{f} : X \rightarrow \mathbb{R}$ such that $f = \rho \circ \tilde{f}$ (this is using the fact that $\mathbb{R}$ is the "universal cover" of S^1).

Since any continuous map homomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ is of the form $t \mapsto rt$ (by Exercise 1.F), it follows any continuous homomorphism $\mathbb{R} \rightarrow S^1$ must be of the form

$$t \mapsto e^{2\pi i r t},$$

for some $r \in \mathbb{R}$. Note multiplying two such characters together corresponds to adding the coefficients r , so we get $\widehat{\mathbb{R}} \cong \mathbb{R}$.

EXERCISE 2.B (Dual group of S^1 is $\mathbb{Z}$). Here you may use the fact (deduced in the Example above) that any continuous homomorphism $\mathbb{R} \rightarrow S^1$ is of the form $t \mapsto e^{2\pi i r t}$, for some $r \in \mathbb{R}$.

- (a) Let $\chi : S^1 \rightarrow S^1$ be a continuous group homomorphism. Show the exponential map $\rho : \mathbb{R} \rightarrow S^1$ given by $t \mapsto e^{2\pi i t}$ induces a continuous homomorphism $\gamma : \mathbb{R} \rightarrow S^1$ by sending

$$\gamma : t \mapsto \chi \circ \rho(t).$$

- (b) Recall the fact (stated in the previous Example) that any homomorphism $\gamma : \mathbb{R} \rightarrow S^1$ is of

the form

$$t \mapsto e^{2\pi i r t}$$

for some $r \in \mathbb{R}$. Use this and the fact that $\chi(1) = \gamma(1) = 1$ to deduce r must be an integer in the case of our constructed γ .

(c) Deduce χ must be of the form $z \mapsto z^r$, for the r above, and conclude $\widehat{S^1} \cong \mathbb{Z}$.

EXERCISE 2.C. Show that $\widehat{\mathbb{Z}} \cong S^1$ (**Hint:** any homomorphism from $\mathbb{Z}$ is entirely determined by the image of 1).

Combining the above two exercises, we deduce that

$$\widehat{\widehat{S^1}} \cong S^1, \quad \widehat{\widehat{\mathbb{Z}}} \cong \mathbb{Z}.$$

This is our first glimpse of the main theorem of Pontryagin Duality.

WARNING. The isomorphisms we have shown in the above two exercises are purely *algebraic*; we have not yet discussed how to turn $\widehat{G}$ into a topological group. We will see that later this section.

EXERCISE 2.D. Show that if G, H are topological groups then show we have the following (algebraic!) isomorphism:

$$\widehat{G \times H} \cong \widehat{G} \times \widehat{H}.$$

EXAMPLE 2.3. By now you will have seen that the torus $\mathbb{T}$ can be expressed as the direct product of the circle with itself. Thus we get

$$\widehat{\mathbb{T}} = \widehat{S^1 \times S^1} = \widehat{S^1} \times \widehat{S^1} = \mathbb{Z}^2$$

EXERCISE 2.E. What is the dual group of $\mathbb{C}^\times$ (the multiplicative group of non-zero complex numbers)?

We want to put a topology on $\widehat{G}$ that tells us if two functions are "close together". For a function $f \in \widehat{G}$, suppose we take a compact set $K \subseteq G$, and an open neighborhood $V \supseteq f(K)$ of the image of this compact set. Since S^1 is Hausdorff, $f(K)$ is a closed set. As such, we might expect that if we wiggle $f(K)$ around a little, it should still be inside V . Thus we want to characterize a function g as being "close" to f if $g(K) \subseteq V$ as well. This motivates the following construction.

DEFINITION 2.4. The **compact-open topology** on $\widehat{G}$ is the one generated by the following neighborhood base of the trivial character:

$$\left\{ W(K, V) := \left\{ \chi \in \widehat{G} : \chi(K) \subseteq V \right\} \right\}_{K, V}$$

where K ranges over all compact sets in G and V ranges over all open neighborhoods of the identity in S^1 (recall it suffices to define constructions on topological groups on neighborhoods of the identity!).

In other words, an open neighborhood of $\widehat{G}$ is a set of characters whose values on some compact set are all close to one another.

KEY IDEA. A sequence of characters $\{\chi_n\}$ in $\widehat{G}$ converge to $\chi \in G$ under the compact-open topology if and only if $\chi_n|_K \rightarrow \chi|_K$ uniformly (as complex-valued functions) on every compact set K .

EXERCISE 2.F (The Dual is a Topological Group). Show that $\widehat{G}$ with the compact-open topology is a topological group. [**Hint:** It suffices to show that if $f_n \rightarrow f \in \widehat{G}$ and $g_n \rightarrow g \in \widehat{G}$ on all compact sets of G then $f_n g_n^{-1} \rightarrow f g^{-1}$ on all compact sets of G .]

A useful notation for open neighborhoods of the identity in S^1 will be the following. For some $\epsilon > 0$, we define

$$N(\epsilon) := \left\{ z \in S^1 : |1 - z| < \epsilon \right\},$$

i.e the intersection of an open ball of radius ϵ around the identity with S^1 . Since S^1 is a topological group, by 1.4 any open set looks like a translation of some $N(\epsilon)$. This will allow us to characterize whether a homomorphism $\chi : G \rightarrow S^1$ is continuous or not.

LEMMA 2.5. Let G be an abelian topological group. A group homomorphism $\chi : G \rightarrow S^1$ is continuous if and only if $\chi^{-1}(N(1))$ is an open neighborhood of the identity $e \in G$.

REMARK. ‘The following proof is not *super* necessary to understand in full detail. The key idea is that (by translation invariance from proposition 1.4) as long as the pre-image of SOME open neighborhood of S^1 is open, then that is actually sufficient to say the entire map is continuous!

Proof. For one direction, if χ is continuous, then the pre-image of any open set must be open, so $\chi^{-1}(N(1))$ has to be an open neighborhood of the identity (since χ was assumed to be a group homomorphism as well, so $\chi^{-1}(1) = e \in G$).

For the other direction, suppose $U := \chi^{-1}(N(1))$ is an open neighborhood of the identity of G . We will prove formulation (5) in Theorem 0.7; that for every $g \in G$ and every open neighborhood $V \ni \chi(g)$, we have a neighborhood $W \ni g$ such that $\chi(W) \subseteq V$. By 1.E, it suffices to just check this for $g = e$

the identity. In other words, for any $\epsilon > 0$ we want to find some $W \ni e$ such that $\chi(W) \subseteq N(\epsilon)$. By continuity of multiplication for any $m \in \mathbb{Z}_{>0}$ we can always find some open neighborhood V such that

$$V^{(m)} := \{v_1 \cdot v_2 \cdots v_m : v_i \in V\} \subseteq U.$$

Thus $\chi(V^{(m)}) \subseteq \chi(U) = N(1)$. Then we use the following observation:

$$\chi(V^{(m)}) \subseteq N(1) \implies \chi(V) \subseteq N(1/m)$$

(feel free to handwave this for now, the point is that multiplication of non-identity elements close to 1 causes the norm to grow). Thus picking m so that $1/m < \epsilon$, we have $\chi(V) \subseteq N(\epsilon)$, and thus χ is continuous. $\square$

THEOREM 2.6. Let G be an abelian topological group. Then:

- (a) if G is compact then $\widehat{G}$ is discrete.
- (b) If G is discrete, then $\widehat{G}$ is compact.
- (c) if G is locally compact then $\widehat{G}$ is locally compact.

Proof. The proofs of part (c) is quite involved and requires some higher-level ideas and techniques we don't have time to cover, but hopefully the statement itself is clear enough. Parts (a) and (b) will be exercises :) $\square$

EXERCISE 2.G (G compact $\implies \widehat{G}$ discrete). Suppose G is compact. Then the subset

$$W(G, N(1)) := \{\chi \in \widehat{G} : \chi(G) \subseteq N(1) \subseteq S^1\}$$

is by definition open in the compact-open topology on $\widehat{G}$. Show that no subgroup of S^1 is contained in $N(1)^a$, and therefore $W(G, N(1))$ contains only the trivial character. Conclude $\widehat{G}$ is discrete.

^aThis is known as the "**no small subgroups**" fact.

EXERCISE 2.H (G discrete $\implies \widehat{G}$ compact). (a) Show the only compact subsets of G are finite subsets, and therefore $f_n \rightarrow f \in \widehat{G}$ if and only if $f_n(g) \rightarrow f(g)$ for all $g \in G$ (i.e the functions converge pointwise).

(b) Consider the space $(S^1)^G := \{g \in G\} S^1$ (i.e the direct product of the circle with itself indexed by elements of G). Explain how elements of $(S^1)^G$ correspond to functions $f : G \rightarrow S^1$. Show $f_n \rightarrow f$ in the product topology of $(S^1)^G$ if and only if $f_n \rightarrow f$ in the compact open

topology of $\widehat{G}$.

(c) Show that $\widehat{G}$ is a closed subset of $(S^1)^G$.

(d) **Tychonoff's theorem** says that the arbitrary direct product of compact topological spaces is compact. Using this fact and the fact that a closed subset of a compact set is compact, deduce $\widehat{G}$ must be compact.

EXERCISE 2.I. Now assume G is an LCA group (in particular recall it is Hausdorff!). Show that $\widehat{G}$ is also Hausdorff. [**Hint:** if $\chi_1 \neq \chi_2 \in \widehat{G}$, we want to find open sets $W(K_1, V_1)$ and $W(K_2, V_2)$ that are disjoint. Picking some $g \in G$ such that $\chi_1(g) \neq \chi_2(g)$, and let $K_1 = K_2 = \{g\}$, and find disjoint open neighborhoods of $\chi_1(g)$ and $\chi_2(g)$.]

COROLLARY 2.7. If G is an LCA group, then so is $\widehat{G}$.

Proof. Combine part (c) of Theorem 2.6 with Exercise 2.I. □

Section 3

(Minimal) Measure Theory

THEOREM 3.1. For any locally compact (Hausdorff!) topological group G , there exists a unique (up to scaling) *translation-invariant* regular measure μ , i.e. $\mu(A + g) = \mu(A)$ for all measurable sets A and $g \in G$. This is called the **Haar measure**.

Proof. Not necessary for us here right now. □

REMARK. Recall if a measure μ is **regular**, then the measure of any measurable set can be approximated by compact sets, i.e

$$\mu(A) = \sup \{ \mu(K) : K \text{ compact and } K \subseteq A \}.$$

EXERCISE 3.A. Show the measure of any non-empty open set $V \subseteq G$ is positive. [**Hint:** Argue via contradiction. Use the fact that you can cover any set with translates of V , and then use the regularity of the measure]

DEFINITION 3.2. For $p \in [1, \infty)$, define $L^p(G)$ as the space of all complex-valued Borel functions on G such that

$$\int_G |f|^p \, dx < \infty.$$

The L^p **norm** is given by

$$\|f\|_p := \left(\int_G |f|^p \, dx \right)^{1/p}.$$

1

EXERCISE 3.B. Show that $L^p(G)$ is a vector space for every p .

EXERCISE 3.C. For f a Borel function on G , if we define $f_y(x) := f(x + y)$, show that $\|f\|_p = \|f_y\|_p$ for all $y \in G$.

DEFINITION 3.3. For any Borel functions f, g on G we define their **convolution** by

$$f * g(x) := \int_G f(x - y)g(y) \, dy.$$

This exists so long as $\|f * g\|_1 < \infty$.

EXERCISE 3.D (Properties of Convolution). Let f, g, h be Borel functions on G .

- (a) Show that if $f * g(x)$ exists, then $f * g(x) = g * f(x)$.
- (b) Show that if $f, g \in L^1(G)$ then $f * g(x)$ exists for almost $x \in G$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
[**Hint:** Consider the function $\phi(x) := \int |f(x - y)g(y)| \, dy$, and use the fact that $|f * g(x)| \leq \phi(x)$ ^a]
- (c) If $f, g, h \in L^1(G)$, show that $f * (g * h) = (f * g) * h$.
- (d) Show that $f * (g + h) = f * g + f * h$.
- (e) Deduce that, if we define multiplication as convolution, then $L^1(G)$ is actually a ring (in addition to being a vector space)! In fancy words, we can call $L^1(G)$ a **Banach algebra**.

^athis is essentially the triangle inequality

Section 4

The Fourier Transform

Throughout we let G denote an LCA group.

DEFINITION 4.1. Given $f \in L^1(G)$, its **Fourier transform** is the function on $\widehat{G}$ given by

$$\widehat{f}(\chi) := \int_G f(x) \chi^{-1}(x) \, dx .$$

EXAMPLE 4.2. We have seen in Example 2.2 that $\widehat{\mathbb{R}} \cong \mathbb{R}$, and that any character of $\mathbb{R}$ is of the form

$$t \mapsto e^{irt}$$

for some $r \in \mathbb{R}$. If we any such character by χ_r , the Fourier transform of any absolutely-integrable function f is given by

$$\widehat{f}(\chi_r) = \int_{\mathbb{R}} f(x) e^{-irx} \, dx .$$

By abuse of notation, this is often written as $\widehat{f}(r)$.

KEY IDEA. When one first learns about Fourier transforms of such functions as above, we think of it as simply a different function on $\mathbb{R}$. However it is *actually* a function on the dual space of $\mathbb{R}$. It just so happens that this is also (algebraically) isomorphic to $\mathbb{R}$. However, we also have to check this isomorphism is topological.

EXERCISE 4.A. Show that the compact-open topology on $\widehat{\mathbb{R}}$ (which is algebraically the same as $\mathbb{R}$) is the same as the Euclidean topology on $\mathbb{R}$. (**Hint:** it suffices to show the neighborhood base of $0 \in \widehat{\mathbb{R}}$ consists of all possible open balls centered at $0 \in \mathbb{R}$).

PROPOSITION 4.3. For any $f \in L^1(G)$, its Fourier transform $\widehat{f}$ is a continuous complex-valued on $\widehat{G}$ (with respect to the compact-open topology).

Proof. Bit too involved for us, but we may take it to be true on faith for now. □

EXERCISE 4.B. Show the following.

- (a) The Fourier transform of $f + g$ is $\widehat{f} + \widehat{g}$.
- (b) The Fourier transform of $f * g$ is $\widehat{f} \cdot \widehat{g}$.

(c) Given $f \in L^1(G)$ and $\chi \in \widehat{G}$, we have $f * \chi(x) = \chi(x)\widehat{f}(\chi)$.

The above results show that the Fourier transform is actually a *ring homomorphism* of the Banach algebra $L^1(G) \rightarrow \mathcal{C}(\widehat{G})$ (where the codomain denotes the ring of continuous functions on $\widehat{G}$).

PROPOSITION 4.4. If $\mathcal{F} : L^1(G) \rightarrow \mathcal{C}(\widehat{G})$ denotes the Fourier transform, then $\mathcal{F}(L^1(G))$ is contained in $\mathcal{C}_0(\widehat{G})$ (the set of continuous functions whose value outside a given compact set is arbitrarily small^a). If we give $\mathcal{C}_0(\widehat{G})$ the topology induced by the supremum metric (i.e $d(f, g) := \sup_{\chi \in \widehat{G}} |f(\chi) - g(\chi)|$) then $\mathcal{F}(L^1(G))$ is dense in $\mathcal{C}_0(\widehat{G})$.

^awe say these functions *vanish at infinity*

Proof. Outside of our scope for now. □

KEY IDEA. To put the above proposition in simple(r) language, it says that almost every continuous function on $\widehat{G}$ that "vanishes at infinity" is arbitrarily close to the Fourier transform of an integrable function on G . This will be useful in the ensuing duality proof.

DEFINITION 4.5. A Borel function $f : G \rightarrow \mathbb{C}$ is **positive definite** if the inequality

$$\sum_{n,m \in [1,N]} c_n \overline{c_m} f(x_n - x_m) \geq 0$$

holds true for all possible tuples $x_1, \dots, x_N \in G$ and $c_1, \dots, c_N \in \mathbb{C}$ for any possible N .

EXAMPLE 4.6 (Positive-Definite Function). Take any $f \in L^2(G)$, (i.e $\int_G |f|^2 dx < \infty$), and define $\widetilde{f}(x) := \overline{f(-x)}$. Then

$$g(x) := f * \widetilde{f}(x)$$

is continuous (this fact you can assume without proof) and positive-definite. To show the latter fact,

for any tuples $\{x_1, \dots, x_N\}$ and $\{c_1, \dots, c_N\}$, we have

$$\begin{aligned}
 \sum_{n,m \in [1,N]} c_n \overline{c_m} g(x_n - x_m) &= \sum_{n,m \in [1,N]} c_n \overline{c_m} \int_G f(x_n - x_m - y) \overline{f(-y)} \, dy \\
 &= \sum_{n,m \in [1,N]} c_n \overline{c_m} \int_G f(x_n - y) \overline{f(x_m - y)} \, dy \\
 &= \int_G \sum_{n,m \in [1,N]} c_n f(x_n - y) \overline{c_m f(x_m - y)} \, dy \\
 &= \int_G \left(\sum_{n \in [1,N]} c_n f(x_n - y) \right) \overline{\left(\sum_{m \in [1,N]} c_m f(x_m - y) \right)} \, dy \\
 &= \int_G \left| \sum_{n \in [1,N]} c_n f(x_n - y) \right|^2 \, dy \\
 &\geq 0
 \end{aligned}$$

THEOREM 4.7 (Bochner). If f is a finite linear combination of positive-definite functions on an LCA group G , then there exists a finite measure μ on $\widehat{G}$ ^a, such that

$$f(x) = \int_{\widehat{G}} \chi(x) \, d\mu.$$

Note that in the above we are consider $\chi \mapsto \chi(x)$ as a function on $\widehat{G}$.

^awhich is *not* the Haar measure!

Proof. Nah. □

The expression on the RHS of the equation in the above theorem can be thought of as a "Fourier transform" of a finite measure μ on $\widehat{G}$. Essentially what this means is that every finite linear combination of continuous positive definite functions can be associated to a finite measure on $\widehat{G}$.

DEFINITION 4.8. Denote by $B(G)$ the set of all finite linear combinations of continuous positive definite functions on G . By Theorem 4.7, these are exactly the functions representable as

$$f(x) = \int_{\widehat{G}} \chi(x) \, d\mu,$$

for some finite measure μ on $\widehat{G}$. We also define

$$B^1(G) := B(G) \cap L^1(G),$$

i.e all (linear combinations of) continuous positive definite functions that are also absolutely integrable.

THEOREM 4.9 (Fourier inversion). If we fix a Haar measure $\mathrm{d}x$ on G , then we can normalize the Haar measure $\mathrm{d}\chi$ on $\widehat{G}$ so that

$$f(x) = \int_{\widehat{G}} \widehat{f}(\chi) \chi(x) \mathrm{d}\chi$$

for all $f \in B^1(G)$.

Proof. See the following few exercises. □

EXERCISE 4.C. For $f \in B^1(G)$, let μ_f denote the finite measure on G so that

$$f(x) = \int_{\widehat{G}} \chi(x) \mathrm{d}\mu_f.$$

Given $f, g \in B^1(G)$ and $h \in L^1(G)$, show that

$$\int_{\widehat{G}} \widehat{h} \widehat{g} \mathrm{d}\mu_f = \int_{\widehat{G}} \widehat{h} \widehat{f} \mathrm{d}\mu_g.$$

. This implies that the measures $\widehat{f} \mathrm{d}\mu_g$ and $\widehat{g} \mathrm{d}\mu_f$ are the same!

Now let $\mathcal{C}_c(\widehat{G})$ denote the vector space of **compactly supported complex-valued continuous functions** on $\widehat{G}$. For any $\phi \in \mathcal{C}_c(\widehat{G})$, let $K_\phi \subseteq \widehat{G}$ be its support (i.e the compact set on which it is nonzero). By Proposition 4.4, we know that the image of the Fourier transform is dense in $\mathcal{C}_c(\widehat{G}) (\subseteq \mathcal{C}_0(\widehat{G}))$. Thus there exists some $f_\phi \in L^1(G)$ such that $\widehat{f}_\phi > 0$ on K_ϕ (think intuitively about why!). We now define a linear functional (i.e a linear map to $\mathbb{C}$) on $\mathcal{C}_c(\widehat{G})$ by

$$\begin{aligned} T : \mathcal{C}_c(\widehat{G}) &\rightarrow \mathbb{C} \\ \phi &\mapsto \int_{\widehat{G}} \frac{\phi}{\widehat{f}_\phi} \mathrm{d}\mu_{f_\phi} \end{aligned} \tag{1}$$

SANITY CHECK. Verify that T is a linear functional and well-defined [**Hint:** For well-definedness, note that Exercise 4.C tells us that the measures $\widehat{f} \mathrm{d}\mu_g$ and $\widehat{g} \mathrm{d}\mu_f$ are the same!]

EXERCISE 4.D. Show that the linear functional T defined in Eqn (1) is **translation-invariant**, i.e.

for any $\chi_0 \in \widehat{G}$ and $\phi \in \mathcal{C}_c(\widehat{G})$, if we define

$$\phi_0(\chi) := \phi(\chi - \chi_0),$$

then $T(\phi) = T(\phi_0)$. **Hint:** define $f(x) := \chi_0(-x)g(x)$, and compare $\widehat{f}$ and μ_f to $\widehat{g}$ and μ_g .

THEOREM 4.10. For any locally compact Hausdorff space X , any linear functional $T : \mathcal{C}_c(X) \rightarrow \mathbb{C}$ induces a measure μ on X such that

$$T(\phi) = \int_X \phi(x) \, d\mu$$

for all $\phi \in \mathcal{C}_c(X)$.

Exercise 4.D, along with Theorem 4.10, tells us that there exists a Haar measure (because T was translation invariant!) $d\chi$ on $\widehat{G}$ such that

$$T(\phi) = \int_{\widehat{G}} \phi(\chi) / \widehat{f_\phi}(\chi) \, d\mu_{f_\phi} = \int_{\widehat{G}} \phi(\chi) \, d\chi.$$

EXERCISE 4.E. By considering $T(\phi\widehat{f})$, show that the measures $\widehat{f}d\chi$ and $d\mu_f$ are the same. Conclude that the Fourier inversion theorem holds.

EXAMPLE 4.11. Consider $G = S^1$, and take $f \in B^1(S^1)$. Write out an explicit formula for the Fourier inversion theorem. What about when $G = \mathbb{T}^2$?

Section 5

Pontryagin Duality

Let G be an LCA group. For any $x \in G$, consider the following map.

$$\begin{aligned}\gamma_x : \widehat{G} &\rightarrow S^1 \\ \chi &\mapsto \chi(x).\end{aligned}\tag{2}$$

EXERCISE 5.A. Show that $\gamma_x \in \widehat{\widehat{G}}$, i.e γ_x is a continuous homomorphism $\widehat{G} \rightarrow S^1$. [**Hint:** for continuity, use the fact that $\{x\}$ is a compact set to show the preimage of any open neighborhood (of the identity) is open.]

Now consider the map

Consider the map

$$\begin{aligned}\alpha : G &\rightarrow \widehat{\widehat{G}} \\ x &\mapsto \gamma_x.\end{aligned}\tag{3}$$

THEOREM 5.1 (Pontryagin Duality). For any LCA group G , the map $\alpha : G \rightarrow \widehat{\widehat{G}}$ given in Eqn. (3) is a homeomorphic isomorphism of topological groups.

REMARK (Terminology). In the above theorem statement, the word "isomorphism" refers to the *group* structure of G and "homeomorphism" refers to the *topological* structure of G . Sometimes the phrase "isomorphism of topological groups" implicitly implies the homeomorphic part as well, but we distinguish these for clarity.

EXERCISE 5.B (α is a homomorphism). Show that the map $\alpha : G \rightarrow \widehat{\widehat{G}}$ is a group homomorphism.

With this established, it suffices to prove three more properties of α to show Pontryagin duality:

1. α is injective. This will show G and $\alpha(G)$ are isomorphic as groups.
2. $\alpha(G)$ is a homeomorphism onto its image. This will show G and $\alpha(G)$ are homeomorphic as topological spaces.
3. $\alpha(G)$ is closed and dense in $\widehat{\widehat{G}}$. This will prove α is a surjection onto $\widehat{\widehat{G}}$, which, combined with the previous parts, shows it must be an isomorphic homeomorphism between G and $\widehat{\widehat{G}}$.

SANITY CHECK. Why a subset of a topological space being closed and dense imply it must be the entire space?

First we will show that α is injective. For any open neighborhood V of the identity of G , we can pick a compact neighborhood W of the identity so that $W - W \subseteq V$. If we define the function

$$f(x) = \begin{cases} 1, & x \in W \\ 0, & x \notin W \end{cases}$$

Then, following Example 4.6, the function

$$g(x) := f * \tilde{f}(x) \tag{4}$$

is continuous and positive definite, and has support contained in $W - W \subseteq V$ (check this!). In particular, $g \in B^1(G)$.

EXERCISE 5.C (α is injective). For the following exercises, let g be the function as in Eqn. (4).

- (a) Suppose we have some $y \in G$ such that $\chi(y) = 1$ for all $\chi \in \widehat{G}$. For any $h \in L^1(G)$, if we define

$$h_y(x) := h(x - y),$$

then show $\widehat{h} = \widehat{h}_y$, as functions in $\mathcal{C}_0(\widehat{G})$. Deduce $h = h_y$ for all $h \in B^1(G)$.

- (b) If $y \neq e \in G$, use part (a), along with the existence and properties of the function g defined in Eqn. (4), to show (via contradiction) that there in fact must exist some $\chi \in \widehat{G}$ so that $\chi(y) \neq 1$. [**Hint:** Use the Hausdorff property of G to pick an appropriate neighborhood V of the identity $e \in G$, and show that g cannot equal g_y .]

- (c) Conclude that α is injective.

EXERCISE 5.D (α is a homeomorphism onto its image). Let $C \subseteq \widehat{G}$ be a compact neighborhood of the trivial character and $\epsilon > 0$, and consider

$$\begin{aligned} V(C, \epsilon) &:= \{x \in G : \chi(x) \in N(\epsilon) \ \forall \chi \in C\} \\ W(C, \epsilon) &:= \left\{ \gamma \in \widehat{\widehat{G}} : \gamma(\chi) \in N(\epsilon) \ \forall \chi \in C \right\}. \end{aligned}$$

We have seen (both by definition of the compact-open topology and the previous exercise) that these form a neighborhood base for the identities in their respective groups.

Show that $\alpha(V(C, \epsilon)) = W(C, \epsilon) \cap \alpha(G)$. Deduce that α is a homeomorphism onto its image.

- EXERCISE 5.E ($\alpha(G)$ is closed). (a) Show that any compact subset K of a Hausdorff topological space must be closed. [**Hint:** for any $x \notin K$, construct an open neighborhood U_x disjoint from K]
- (b) Show that any locally compact dense subset A of a Hausdorff space X is open. [**Hint:** A compact neighborhood of $a \in A$ is the same as $\overline{U} \cap A$ for some U open in X . Show that $U \setminus (\overline{U} \cap A)$ is empty.]
- (c) Use the above, along with Exercise 5.D to show that $\alpha(G)$ is closed. [**Hint:** Use the fact that $\alpha(G)$ is dense in $\overline{\alpha(G)}$, and part (b) of Exercise 1.G]

PROPOSITION 5.2. The image of α in $\widehat{\widehat{G}}$ is dense.

Proof. We don't have time to cover this during SUMaC, but this has to do with the Plancherel Formula over arbitrary LCA groups. Ask me later if you're still curious, but shouldn't be put in the presentation either way. $\square$

The above proposition, along with the preceding exercises, finishes our proof of Pontryagin duality!