

HELLO, ADELE: TATE'S THESIS AND ADELIC ANALYSIS

BY
AARYAN SUKHADIA

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Chapter

INTRODUCTION

OVERVIEW

Adeles have gone by various names over the course of mathematical history. A term first introduced by Chevalley as a wordplay on ideals [Che40], they have been referred to as the "ring of valuation vectors" by John Tate in his seminal thesis [Tat50], "repartitions" by Jean-Pierre Serre [Ser88], before eventually the original term¹ stuck around. Whatever one may choose to call them privately, they have now become a staple of the language of the modern number theorist, and this thesis is an exploration of them.

Chapter I covers the motivations behind and basic properties of the restricted product to get the reader familiar with the adelic language.

Chapter II recounts the substance of Tate's famous PhD thesis, which proves the classical Hecke functional equation between zeta functions of global fields using an adelic formulation. We follow primarily the exposition of [RV98], [Wei74] and Tate's thesis itself [Tat50], modernizing the notation as we go along.

None of the material covered in this thesis is novel, recent, or even niche mathematics, and there are a plethora of sources on this; the References section of this thesis likely contains but a tiny fraction of them. To me this thesis is two things primarily: (1) a process by which I teach and familiarize myself with this field of mathematics via written exposition, and (2) a celebration of this elegant algebraic object that has become so ubiquitous in number theory. If for some inexplicable reason you, the reader, are someone distinct from myself or my advisor, I hope you find beauty and fascination galore from reading about this stuff. I know I sure did writing it.

¹This is a slight simplification; Chevalley originally only came up with the multiplicative ideles, it was Weil who generalized the additive adeles and ended up popularizing this name.

FORMAT, BACKGROUND, AND CONVENTIONS

Format

In the days of old, mathematics was written densely and read in black and white on paper due to cost and typesetting considerations. With the wonders of modern technology I have chosen to make use of a variety of boxes and colors throughout this text because it makes it far less painful to read, review, and skim this work. The three main classes of boxed environments used will be as follows:

DEFINITION 0.0.1. A bright, almost-blaring red-backgrounded, red-striped box to house a vital DefinitionTM.

EXAMPLE 0.0.2. An amber-striped environment to indicate an important or prototypical ExampleTM of a concept is contained in its confines.

THEOREM 0.0.3. A neutrally-toned gray-shaded, box to house an important result, whether that be a TheoremTM (such as this particular box), a PropositionTM, etc.

As the reader might have deduced, Definitions, Examples and Results follow a [Subsection.Number] numbering scheme, all using a common counter. Equations throughout will follow a (Section-Number) numbering scheme.

As is standard in mathematical writing I've also filled this text with technically unnecessary but hopefully-enriching RemarksTM. Less-standard variants of asides I've used here are Sanity ChecksTM (statements that should be intuitively obvious and verifiable in some epsilon time to an attentive reader) and WarningsTM (cautionary interventions about potentially confusing or unintuitive notation, terminology or concepts).

Assumed Background and Conventions

A writer of mathematics is always challenged with balancing mathematical accessibility with the limits of time and space. As such, we assume some familiarity with measure theory, functional analysis, algebraic number theory and algebraic geometry, all at an undergraduate level. As a litmus test we suggest the reader go through the following subsection on Conventions and Notations. The goal of this text is that it should be broadly understandable to a reader comfortable with some dense subset of the conventions described below.

Miscellaneous Notation Conventions

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ will denote the sets of Naturals, Integers, Rationals, Reals and Complex Numbers respectively. Naturals include 0.
- S^1 denotes the circle, i.e the group of norm-1 complex numbers.
- $\mathbb{F}_q$ denotes the (unique, up to isomorphism!) finite field with q elements. In general by $\text{char}(K)$ we denote the **characteristic** of K .

- When we have an infinite product or a vector with infinitely many components we say "**almost everywhere**" to mean "in all but finitely many components". If the components of the infinite product are indexed by $i \in I$ then we write $(x_i)_i$ to mean the "vector" of a single element in the infinite product, and x_i to mean the i -th coordinate.
- For a subset A of a set X , we denote by A^c or $X \setminus A$ the complement of A in X .
- Given a function f defined on a space X and a subset $Y \subseteq X$, we write $f|_Y$ for the restriction of f to Y .
- We recall the complex-valued **gamma function** $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$, a priori defined (and convergent!) on $\Re(z) > 0$ and extends to $\mathbb{C}$ via meromorphic continuation with poles at integers $k \leq 0$.

Measures and Topological Groups

- The phrase "**almost everywhere**" means we are considering a set whose complement has measure zero.
- By a **locally compact group** G we mean G is Hausdorff and every point has a compact neighborhood. Note we will *always* assume the Hausdorff hypothesis. In many cases the locally compact groups we work with are also assumed to be abelian. We use the abbreviation "**LCA group**" in such cases.
- Recall that every locally compact group admits a (left-invariant) **Haar measure**, which we often denote by dx or dy or dz , and so on. This will be a slight abuse of notation since we may change the variable depending on what is being integrated, but before any such integration we explicitly describe the Haar measure. Given a Borel (i.e measurable) set $A \subseteq G$, we denote by $\text{Vol}(A, dx)$ the measure of A with respect to dx .
- For G an LCA group, we denote by $\widehat{G}$ the **Pontryagin dual group** of continuous characters $G \rightarrow S^1$. We will assume the standard classical results of Pontryagin Duality for an LCA group, and refer the reader [Rud91] for a detailed exposition on them.

Algebra and Number Theory

- Given subgroups H, K of a commutative group G with operation denoted by addition, we denote by $H + K$ the subgroup of all elements of the form $h + k$ with $h \in H$ and $k \in K$.
- All rings are assumed to be commutative and unital; $R^\times$ will denote the set of invertible elements of a ring R .
- We use $H \leq G$ and $I \trianglelefteq R$ to denote " H is a subgroup of the group G " and " I is an ideal of the ring R " respectively. If $r_1, r_2, \dots$ is a set of elements in a group G (resp. a ring R), we denote by $\langle r_1, r_2, \dots \rangle$ the subgroup (resp. ideal) generated by the r_i .
- The symbol $\mathfrak{p}$ *always* refers to a prime ideal, and in the case that we are working in a discrete valuation ring we denote a choice of uniformizer by π . When necessary we add subscripts to indicate the relevant field we are working in. When working in a Dedekind domain we denote by $N\mathfrak{a}$ the **norm** of a nonzero ideal $\mathfrak{a}$, i.e the cardinality of $R/\mathfrak{a}$.

- By a **local field** we mean a (non-trivially!) valued field whose additive group is locally compact with respect to the topology induced by the valuation. Note this convention includes $\mathbb{R}$ and $\mathbb{C}$, though they are sometimes excluded from the definition in other texts.
- We say a local field F is **Archimedean** if it is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and **non-Archimedean** otherwise. For F non-Archimedean with absolute value $|\cdot|_F$ we denote by $\mathcal{O}_F$ its **valuation ring** (i.e the ring of elements with absolute value ≤ 1).
- By a **global field** K we mean a finite extension of $\mathbb{Q}$ (in which case we call K a **number field**) or of $\mathbb{F}_p(t)$ (in which case we call K a **function field**). For any (non-trivial!) valuation v on K we denote by K_v the **completion** of K with respect to v ; this is always a local field. When K_v is non-Archimedean we write $\mathcal{O}_v$ to denote the valuation ring of K_v . Note this convention particularly in the case of $K = \mathbb{Q}$; when we write $\mathbb{Z}_p$ we are *always* referring to the p -adic integers! We assume the reader is familiar with the standard fact that every local field arises as the completion of a global field.
- We denote by $\mathcal{P}_K$ the set of all **places** (i.e topological equivalence classes of absolute values) of a global field K . We may speak of a place v being Archimedean, non-Archimedean, real, complex, if K_v is Archimedean, non-Archimedean, real, complex, respectively. In particular when $\text{char}(K) > 0$ it has no Archimedean places.
- We recall that if K'/K is an algebraic extension of fields than any nontrivial absolute value $|\cdot|_K$ on K extends uniquely to K' . Thus if $v \in \mathcal{P}_K$ and $w \in \mathcal{P}_{K'}$ are such that there exists an absolute value on K corresponding to v whose unique extension corresponds to $w \in \mathcal{P}_{K'}$, then we say w **lies over** v , denoted $w | v$.
- We say a place v of a global field K is **infinite** if it is Archimedean, and **finite** otherwise. In particular, all places of a global function field are finite.

Algebraic Geometry

- An **algebraic variety** X is an integral, separated, finite-type affine scheme over a field k . An **algebraic curve** is a 1-dimensional algebraic variety. The **function field** $k(X)$ is the stalk of the (unique!) generic point of X , or equivalently the field of fractions of $\mathcal{O}_X(U)$ for any non-empty open U .
- Recall the function field of a smooth proper algebraic curve over $\mathbb{F}_q$ (hence projective!) is a global field of positive characteristic. We also assert the converse; that any positive-characteristic global field K in which $\mathbb{F}_q$ is algebraically closed arises as the function field of some smooth projective algebraic curve geometrically connected over $\mathbb{F}_q$ (see §1.6 of [Har77] for a detailed proof, modulo the notion of geometric connectedness).
- For any (separated) k -scheme X with structure morphism $X \rightarrow \text{Spec}(k)$, if we let $\mathcal{I}$ be the ideal of the diagonal subscheme of the fiber product $X \times_{\text{Spec}(k)} X$, we recall the **module of Kähler differentials** $\Omega_{X/k}^1$ is given by $\mathcal{I}/\mathcal{I}^2$ equipped with the k -linear derivation $d : \mathcal{O}_X \rightarrow \mathcal{I}/\mathcal{I}^2$ given by $f \mapsto f \otimes 1 - 1 \otimes f$. This is universal among k -linear derivations from $\mathcal{O}_X$ to quasi-coherent $\mathcal{O}_X$ -modules $\mathcal{F}$, in the sense that any such derivation $\delta : \mathcal{O}_X \rightarrow \mathcal{F}$ uniquely factors through it. Intuitively we may view it as the sheaf of 1-forms on X .

ACKNOWLEDGEMENTS

This thesis, and the continuation of it I hope to write in the future, is a product of the mathematical environment I have been fortunate enough to inhabit these past four years.

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Thirdly, I would like to express a broad appreciation to anyone who has either learned from or taught me mathematically. I enjoy teaching mathematics as much as I do learning it, and the more I do one the more I feel motivated to do the other. So whether you're one of the students I've tutored, a professor I've talked to in office hours, or just a friend who I've talked about mathematics with, thank you.

Finally my family and broader community of friends for supporting me through my time here at university and earlier in my life. I wouldn't be where I am without them.

Chapter I

HELLO, ADELE

*"In retrospect, we see now that the real numbers appear there as . . . neither more nor less interesting to the arithmetician than its p -adic companions, and that there is at least one language and one technique, that of the **adeles**, for bringing them all together under one roof and making them cooperate for a common purpose."*

Preface to *Basic Number Theory*

ANDRÉ WEIL

The adelic language gives a beautiful solution to an age-old problem: that of doing Fourier analysis on a global field K . The classical approach was to pass to a tensor product of K with $\mathbb{R}$ (over $\mathbb{Q}$), but with the discovery of (infinitely many) non-trivial absolute values, it was not too far a stretch to imagine a structure that captured the information of all the completions at once, and "bringing them all together", in the words of Weil.

Instead of taking just the direct product of all the completions, however, we take the *restricted direct product*, which imposes an "almost-everywhere integrality" condition on the elements. The reason for this is twofold: (1) to give the group a locally compact Hausdorff structure, without which we could not do the appropriate harmonic analysis, and (2) to make the natural embedding of K discrete and cocompact, which will be invaluable.

This chapter proceeds in a somewhat reverse historical order. Section 1 recounts the basic results on the adeles and their unit group (the ideles) in somewhat reverse historical order (or rather, one might say history discovered the ideles in the reverse order). We build the theory in this section with pedantry and care, to allow for a full appreciation for the nuances of the adelic formulation.

REMARK. Some texts (e.g. [Ber+13]) use the accented terms "adèles" and "idéles", but Chevalley introduced these terms more as a wordplay on "ideals," and so they don't necessarily exactly correspond to French words. Besides, I've never seen accents used for these terms by a French writer other than Chevalley's original paper (see [Wei74] and [Lan94]), so I have chosen not to accent them in this text either.

SECTION 1

ADELES AND IDELES

§1.1 Introduction: Meeting the Adeles and Ideles

Recall the ring of profinite integers;

$$\widehat{\mathbb{Z}} := \varprojlim_{n \in \mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \prod_{p \text{ prime}} \mathbb{Z}_p,$$

where the inverse limit is taken over the natural maps of quotient rings ordered by divisibility. Imbued with the natural profinite topology, this is a compact Hausdorff space, and thus has all the nice properties that come along with that.

WARNING (Overloaded Notation). Throughout this thesis, for a locally compact Hausdorff topological group G we use $\widehat{G}$ to denote the Pontryagin dual of G . This is visibly not what we mean by $\widehat{\mathbb{Z}}$ above. Unfortunately the notation $\widehat{R}$ is commonly to denote completions of rings, so distinguishing the two is simply a matter of context.

Suppose we wanted to do a similar thing with $\mathbb{Q}$ and all of its completions. We run into a slight issue here, since $\prod \mathbb{Q}_p$ is not locally compact. Local compactness (and Hausdorffness) is rather necessary if one desires to do any sort of analysis with topological groups, since we would like to have a natural Haar measure on our group. Amending this issue is precisely the motivation behind the construction of the *restricted product*, which we dive into now.

Let J be an indexing set such that for every $v \in J$ we have a locally compact Hausdorff group G_v indexed by v , satisfying the property that for J_∞ a finite (possibly empty) subset of J , we have a compact open (and thus closed!) subgroup $H_v \leq G_v$ for all $v \in J_\infty$.

DEFINITION 1.1.1. The **restricted direct product** (also just called the **restricted product**) of the G_v with respect to the H_v is defined as

$$\prod_{v \in J} G_v := \left\{ (x_v) \in \prod_{v \in J} G_v : x_v \in H_v \text{ for all but finitely many } v \right\}.$$

REMARK (Notation). The notation I've used for the restricted product (product and coproduct symbol overlayed on one another) is non-standard, but more distinct than the usual $\prod'$. I first saw this notation in [Sut15].

Let us denote this restricted product as G . Note $G \subseteq \prod G_v$ (the usual direct product of groups). We want to topologize G , but the subspace topology induced by the product topology on $\prod G_v$ is insufficient. For reasons that will become clear later, we'll want subsets like $\prod_{v \in I} G_v \times \prod_{v \notin I} H_v$ to be open for finite $I \supseteq J_\infty$, which won't be true in the subspace topology unless $H_v = G_v$ for all but finitely many $v \in J$.

DEFINITION 1.1.2. The **restricted product topology** on $\prod G_v$ is given by the following neighborhood basis

of $g = (g_v)_v$:

$$\left\{ \prod U_v : U_v \subseteq_{\text{open}} G_v, g_v \in U_v \text{ and } U_v = H_v \text{ for all but finitely many } v \right\},$$

i.e. the (non-empty) open sets around g are exactly the ones that contain H_v in cofinitely many components, and arbitrary open neighborhoods of g_v in the rest.

OBSERVATION. Suppose we have a family of compact open subgroups $H'_v \subseteq G_v$ such that $H'_v = H_v$ for all but finitely many $v \in J$. Then we note that $\prod G_v$ with respect to the H_v is the same (both set-theoretically and topologically) as $\prod G_v$ with respect to the H'_v . It thus suffices to specify the H_v for all but finitely many v . A parallel observation we make is that for any finite positive integer n we have the topological isomorphism:

$$\prod_v G_v^n \cong \prod^n \left(\prod_v G_v \right) \\ (\dots, (g_{v,1}, \dots, g_{v,n}), \dots)_v \mapsto ((g_{v,1})_v, \dots, (g_{v,n})_v), \quad (1.1-1)$$

where, naturally, G_v^n is the n -fold direct-product of G_v with itself.

We introduce a handy piece of notation that will come in useful later. Let S be any finite subset of the index set J containing J_∞ . Then we define

$$G_S := \prod_{v \in S} G_v \times \prod_{v \notin S} H_v, \quad (1.1-2)$$

endowed with the usual product topology. We note, for any S , that this is by definition an open neighborhood of the identity $(e_v)_v$ of G . Moreover, for any finite $T \supseteq S$ consider the inclusion map $i_{S,T} : G_S \hookrightarrow G_T$, which is an open embedding. Then we see that the restricted product G is in fact the directed union of the G_S 's as open subspaces:

$$G = \varinjlim_S G_S. \quad (1.1-3)$$

We leave it to the reader to verify this abstract-nonsense (but ultimately useful and simpler!) characterization, which ultimately comes down to checking the relevant universal property is satisfied.

PROPOSITION 1.1.3. Let G_v, H_v be as defined above and G be the restricted product. Then

- (a) G is a locally compact group.
- (b) A subset $Y \subseteq G$ has compact closure if and only if $Y \subseteq \prod K_v$ for some family of compact subsets $K_v \subseteq G_v$ such that $K_v = H_v$ for all but finitely many v .

Proof. Part (a)

We first verify that it is indeed a topological group. The group structure is clear; for continuity of the group maps, recall any basic open set is of the form

$$\mathcal{U} := \prod_{v \in S} U_v \times \prod_{v \notin S} H_v$$

for some finite indexing set S and opens $U_v \subseteq G_v$. Since each H_v is a subgroup, the preimage of $\mathcal{U}$ under the inversion map is:

$$\prod_{v \in S} (U_v)^{-1} \times \prod_{v \notin S} H_v,$$

which is also open since each G_v is a topological group, thereby giving us continuity of inversion.

For multiplication, the topological isomorphism in (1.1-1) gives us $G \times G \cong \prod_v (G_v \times G_v)$, allowing us to similarly look at preimages component by component. In particular, consider any element

$$\left((k_v, k'_v)_{v \in S}, (k_v, k'_v)_{v \notin S} \right)$$

in the preimage of $\mathcal{U}$ under the group law $m = \prod_v m_v$, defined in the natural way. By definition, away from some finite index set $T \supseteq S$, we must have that $(k_v, k'_v) \in H_v \times H_v$ for all $v \notin T$, and clearly $H_v \times H_v$ is in the pre-image of $\mathcal{U}$. For any $v \in T$, we know from the fact that G_v is a topological group that (k_v, k'_v) has some open neighborhood $W_v \times W'_v$ contained in the pre-image of $\mathcal{U}_v$. Thus:

$$\left((k_v, k'_v)_{v \in S}, (k_v, k'_v)_{v \notin S} \right) \in \prod_{v \in T} (W_v \times W'_v) \times \prod_{v \notin T} (H_v \times H_v) \subseteq m^{-1}(\mathcal{U}),$$

and so multiplication is continuous. Thus G is a topological group, and it remains to show local compactness. Since each H_v is compact, Tychonoff tells us $\prod_{v \notin S} H_v$ is also compact. Thus G_S is the product of finitely many locally compact spaces with a compact space, and is thus itself locally compact. Moreover, the product topology on G_S is the *same* as the subspace topology induced by G , and since any $(x_v) \in G$ is contained in such a G_S , we conclude G must be locally compact as well.

Part (b)

If Y is contained in some product of compact sets $K = \prod K_v$ (which is compact by Tychonoff) its closure $\bar{Y}$ is a closed subset of a compact subset of a Hausdorff space G , and thus $\bar{Y}$ is compact too, proving one direction.

For the other direction, suppose $\bar{Y}$ is a compact subset of G . Since the open sets G_S defined above cover G , a finite number of such G_S must cover $\bar{Y}$. Note a union of a finite number of G_S is just a set of the form $G_{\mathcal{S}}$, for $\mathcal{S}$ the intersection of all the index sets S . Since the projection maps $\rho_v : G \rightarrow G_v$ are continuous, each $\rho_v(\bar{Y}) \subseteq G_v$ is compact. And since all but finitely many components of $G_{\mathcal{S}}$ are equal to H_v , it follows $\rho_v(\bar{Y}) \subseteq H_v$ for all but finitely many v . Thus $\bar{Y}$, and therefore Y , is contained in a set of the desired form. $\square$

WARNING. Our work above seems (and may very well be) a tad pedantic with the details. However it is worth being careful about constructions such as this; one might be tempted to think that if we have *any* directed system of topological groups with inclusion that the colimit (with the colimit topology) must surely be a topological group. The editors of MIT's second-edition *Encyclopedic Dictionary of Mathematics* thought so too (see Article 210, pg. 805). This turns out not to be the case: the colimit of the directed system of $G_i := \mathbb{Q} \times \mathbb{R}^i$ does not have continuous addition. The key issue here is that colimit topologies can fail to commute with taking direct products. However [TSH98] shows that we do get a nice output in this colimit construction when the groups are all locally compact.

OBSERVATION. For each v we have a natural topological embedding $G_v \hookrightarrow G$ into the v -th component of the restricted product. Thus we naturally identify G_v with a closed subgroup of G .

We are now ready to define the adeles. Let K be a global field, and let K_v denote its completion at a place v . With respect to addition, note K_v is a locally compact additive group. For all finite places v , recall K_v admits its local ring of integers $\mathcal{O}_v$ as an open compact subgroup.

DEFINITION 1.1.4. The **adele group** $\mathbb{A}_K$ of a global field K is the restricted direct product of K_v with respect to the open subgroups $\mathcal{O}_v$. Since this admits a natural topological ring structure as well, it is also referred to as the **adele ring**.

Note we have an embedding

$$\begin{aligned} K &\hookrightarrow \mathbb{A}_K \\ x &\mapsto (x, x, \dots) \end{aligned}$$

This is well-defined since K always has a natural embedding in K_v for any place v , and any $x \in K$ is integral (i.e. has absolute value ≤ 1) at almost all places $v \in \mathcal{P}_K$.

Similarly, for each place v we can consider $K_v^\times$, the locally compact group of units of K_v with respect to multiplication. Then the local integral units $\mathcal{O}_v^\times$ are an open compact subgroup of each $K_v^\times$.

DEFINITION 1.1.5. The **idele group** $\mathbb{I}_K$ of a global field K is the restricted direct product of the $K_v^\times$ with respect to $\mathcal{O}_v^\times$.

We have a similar embedding

$$\begin{aligned} K^\times &\hookrightarrow \mathbb{I}_K \\ x &\mapsto (x, x, \dots) \end{aligned}$$

which is again well-defined by a similar argument as with the adeles. This motivates the following definition.

DEFINITION 1.1.6. The elements in the image of the diagonal embedding of K (resp. $K^\times$) into $\mathbb{A}_K$ (resp. $\mathbb{I}_K$) are called the **principal adeles** (resp. **principal ideles**).

WARNING. The notation $\mathbb{I}_K$ and $\mathbb{A}_K^\times$ is used interchangeably in some literature, but one should be careful; although the group $\mathbb{I}_K$ is *algebraically* the same as the group of units $\mathbb{A}_K^\times$, this isn't a topological homeomorphism if we take the naive subspace topology $\mathbb{A}_K^\times \subseteq \mathbb{A}_K$. In general, recall for any topological ring R , we topologize the unit group $R^\times$ by the subspace topology of its image under the injection

$$\begin{aligned} R^\times &\hookrightarrow R \times R \\ r &\mapsto (r, r^{-1}), \end{aligned}$$

where of course the codomain is given the natural product topology. We do this because taking the subspace topology on $R^\times$ from R may fail to make it a topological group, as in the Example below.

EXAMPLE 1.1.7. Take $K = \mathbb{Q}$, and consider the sequence of ideles $(1, 1 + n!, 1 + n!, \dots) \in \mathbb{A}_\mathbb{Q}^\times$, i.e. 1 at the infinite place and $1 + n!$ at all the finite places. For any prime p we have

$$\lim_{n \rightarrow \infty} |1 - (1 + n!)|_p = \lim_{n \rightarrow \infty} p^{-\text{ord}_p(n!)} = 0,$$

and thus this sequence converges to 1 in the subspace topology of $\mathbb{A}_\mathbb{Q}^\times$. If $\mathbb{A}_\mathbb{Q}^\times$ were a topological group, we should expect its reciprocal $(1, 1/(1 + n!), 1/(1 + n!), \dots)$ to also converge to 1, by continuity of inversion, but we show now this is not the case.

To see more explicitly what's happening, take an open neighborhood, in the *subspace topology*, of $1 \in \mathbb{A}_{\mathbb{Q}}^{\times}$. Every such neighborhood has the form

$$U := \left(\mathbb{R} \times \prod_{p \in S} U_p \times \prod_{p \notin S} \mathbb{Z}_p \right) \cap \mathbb{A}_{\mathbb{Q}}^{\times},$$

where S is some finite collection of primes, and U_p is a neighborhood of $1 \in \mathbb{Q}_p$ for each $p \in S$. However, consider the following set:

$$V := \mathbb{R}^{\times} \times \prod_{p < \infty} \mathbb{Z}_p^{\times},$$

which is visibly open in the idelic topology. We claim $U \not\subseteq V$ for any U described above. The reason for the non-inclusion is that for any U we can select some $(x_p)_p \in \mathbb{A}_{\mathbb{Q}}^{\times}$ such that $x_p \in \mathbb{Z}_p \setminus \mathbb{Z}_p^{\times}$ for some $p \notin S$. This thus means $(x_p)_p \notin V$. In particular, this is why the sequence $(1, 1/(1+n!), 1/(1+n!), \dots)$ doesn't converge to 1 in the idelic topology: for no n does that element lie in the set V constructed above, since $1+n!$ is always divisible by *some* prime. Thus we see that the relative open sets are too coarse, i.e. the idelic topology is *finer* than the induced subspace topology.

The natural question, then, is whether the topology on $\mathbb{I}_K$ as the unit group of $\mathbb{A}_K$ is the restricted product topology. The answer, as it neatly turns out, is yes!

PROPOSITION 1.1.8. The subspace topology on $\mathbb{I}_K \subseteq \mathbb{A}_K \times \mathbb{A}_K$ (the image under the embedding $u \mapsto (u, u^{-1})$) is exactly the restricted product topology.

Proof. Suppose $\mathbb{A}_K^{\times}$ is given the idelic topology, and let $\phi : \mathbb{A}_K^{\times} \rightarrow \mathbb{I}_K$ be the natural bijection. It is relatively clear that ϕ is an open map; indeed, for any basic open set in the idelic topology

$$U = \prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v^{\times},$$

we have that the image of U is:

$$\phi(U) = \left(\left(\prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v^{\times} \right) \times \left(\prod_{v \in S} U_v^{-1} \times \prod_{v \notin S} \mathcal{O}_v^{\times} \right) \right) \cap \mathbb{I}_K,$$

which is visibly open in the topology of $\mathbb{I}_K$.

Thus it remains to show ϕ is continuous if $\mathbb{A}_K^{\times}$ is given the idelic topology. Take an open neighborhood U of the multiplicative identity of $\mathbb{A}_K \times \mathbb{A}_K$. Any such neighborhood contains one of the form

$$U = \left(\prod_{v \in S} U_{1,v} \times \prod_{v \notin S} \mathcal{O}_v \right) \times \left(\prod_{v \in T} U_{2,v} \times \prod_{v \notin T} \mathcal{O}_v \right),$$

where S, T are finite indexing sets containing the infinite places. In the following expressions, for any set $S_v \subseteq K_v$ we let $S_v^{\times}$ be shorthand for $S_v \cap K_v^{\times}$. Taking intersection of U with the image of ϕ gives us the intersection of $\mathbb{I}_K$ with the following set:

$$\begin{aligned} U \cap \phi(\mathbb{A}_K^{\times}) &= \prod_{v \in S \cap T} \left(U_{v,1}^{\times} \cap (U_{v,2}^{\times})^{-1} \right) \times \left((U_{v,1}^{\times})^{-1} \cap U_{v,2}^{\times} \right) \\ &\quad \times \prod_{v \in S \setminus (S \cap T)} (U_{v,1} \cap \mathcal{O}_v^{\times}) \times \left((U_{v,1}^{\times})^{-1} \cap \mathcal{O}_v^{\times} \right) \end{aligned}$$

$$\begin{aligned}
& \times \prod_{v \in T \setminus (S \cap T)} \left(\mathcal{O}_v^\times \cap (u_{v,2}^\times)^{-1} \right) \times \left(\mathcal{O}_v^\times \cap u_{v,2}^\times \right) \\
& \times \prod_{v \notin S \cup T} \mathcal{O}_v^\times \times \mathcal{O}_v^\times.
\end{aligned}$$

Note in the above (rather heinous) expression we used our observation in (1.1-1) to "absorb" the direct product of $\mathbb{A}_K$ into the restricted product itself. In any case, pulling back to $\mathbb{A}_K^\times$ corresponds to just looking at the first coordinate in each of the products. We note that for all the $v \in S \cup T$ the set in the v -th component is visibly an open neighborhood of the identity, and since this is still a finite set the pullback is indeed equal to $\mathcal{O}_v^\times$ on cofinitely many components. We deduce it is an open set of the idelic topology, thereby proving ϕ is continuous and thus a homeomorphism onto its image. $\square$

It will be useful to explore the behavior of the adeles under base-change - in particular, extending scalars by to finite separable extensions. This type of technique will allow us to show results for any $\mathbb{A}_K$ by just considering $K = \mathbb{Q}$ or $K = \mathbb{F}_p(t)$.

PROPOSITION 1.1.9 (Adelic Base Change). Take E/K a finite separable extension of global fields, and let $\{u_1, \dots, u_n\}$ be a K -basis of E . Then, recalling we have a canonical isomorphism $E \otimes_K K_v \xrightarrow{\sim} \prod_{w|v} E_w$, the natural map

$$\begin{aligned}
\alpha : \bigoplus_{j=1}^n \mathbb{A}_K &\rightarrow \mathbb{A}_E \\
((x_{v,j})_v)_j &\mapsto \sum_j u_j (x_{v,j})_v
\end{aligned} \tag{1.1-4}$$

is an isomorphism of topological groups.

Proof. For every place v of K , we define:

$$E_v := \prod_{w|v} E_w = E \otimes_K K_v.$$

where the product is taken over all places w of E lying above v , of which there are at most n . This is manifestly a topological algebra over K_v , into which we have a natural diagonal embedding. We also recall the classical fact that E_v admits $\{u_1, \dots, u_n\}$ as a K_v -basis, and thus the map

$$\begin{aligned}
\alpha_v : K_v^{\oplus n} &\xrightarrow{\sim} E_v \\
(x_j) &\mapsto \sum_j u_j x_j
\end{aligned}$$

is an algebraic and topological isomorphism of K_v -vector spaces.

In a similar vein, for $v \nmid \infty$ we can define

$$\mathcal{O}_{E_v} = \prod_{w|v} \mathcal{O}_{E_w},$$

and again the u_j 's define a topological and algebraic isomorphism

$$\alpha_v : \mathcal{O}_{K_v}^{\oplus n} \xrightarrow{\sim} \mathcal{O}_{E_v}$$

for all v away from a finite set S_0 of places containing the Archimedean ones. For any $S \supseteq S_0$, consider the

products $\mathbb{A}_{K,S}$ and $\mathbb{A}_{E,S}$ as defined in (1.1-2). We note then that:

$$(\mathbb{A}_{K,S})^{\oplus n} = \bigoplus_{i=1}^n \left(\prod_{v \in S} K_{v,i} \times \prod_{v \notin S} \mathcal{O}_{K_{v,i}} \right) \cong \prod_{v \in S} E_v \times \prod_{v \notin S} \mathcal{O}_{E_v} =: \mathbb{A}_{E,S}.$$

Thus for any such S , by appropriately restricting the collection $\{\alpha_v\}$ induces a map

$$\alpha_S := (\mathbb{A}_{K,S})^{\oplus n} \xrightarrow{\sim} \mathbb{A}_{E,S}.$$

Recalling our abstract-nonsense equivalent definition of the adeles in (1.1-3), we deduce that the map α , as the colimit of the directed system of isomorphisms under inclusions, is a topological and algebraic isomorphism. $\square$

§1.2 Adelic Characters and Measures

We review the basic notion of Pontryagin dual before continuing.

DEFINITION 1.2.1. Let G be a locally compact topological group.

- A **quasi-character** of G is an element $\chi \in \text{Hom}_{\text{cont}}(G, \mathbb{C}^\times)$.
- A **character** of G is an element $\chi \in \text{Hom}_{\text{cont}}(G, S^1)$.

REMARK. Some texts refer to "quasi-characters" as "characters" and refer to "characters" as "unitary characters". We use the other convention because it is consistent with Tate's formulation in his thesis [Tat50] and because it gives us an excuse to say and write the prefix "quasi".

Pontryagin duality is the necessarily bedrock upon which one builds the theory of Fourier analysis on (locally compact Hausdorff) groups. Our goal will be to show the adeles are in-fact self-dual, hence giving us a nice Fourier transform on this space.

We first begin with the construction of the so-called *standard characters* ψ_F of local fields, as these are the building blocks of the adelic characters. Throughout we denote by F a local field, and in the non-Archimedean case we write $\mathcal{O}_F$ for its (local!) valuation ring with unique maximal ideal $\mathfrak{p}$ and uniformizer π .

- For $F = \mathbb{R}$, we define

$$\psi(x) := e^{-2\pi i x}. \quad (1.2-1)$$

- For $F = \mathbb{Q}_p$, we define our character by the composition of the following maps:

$$\psi : \mathbb{Q}_p \rightarrow \mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}[1/p] / \mathbb{Z} \hookrightarrow S^1,$$

where the first map is the canonical quotient map, the second is the inverse of the map $\alpha + \mathbb{Z} \mapsto \alpha + \mathbb{Z}_p$, and the last is the restriction of the canonical isomorphism $\mathbb{R}/\mathbb{Z} \xrightarrow{\sim} S^1$ via $e^{2\pi i x}$. Calculating this chain of maps, it is not too hard to verify that

$$\psi(x) := e^{2\pi i r_x / p^{n_x}}, \quad (1.2-2)$$

where n_x is a non-negative integer such that $p^{n_x} x \in \mathbb{Z}_p$, and $r_x \in \mathbb{Z}$ represents $p^{n_x} x$ modulo p^{n_x} .

- All the remainder of the characteristic zero cases fall under when F'/F a finite separable extension of any of the above F , so we may define

$$\psi_{F'}(x) := \psi_F(\text{Tr}_{F'/F}(x)). \quad (1.2-3)$$

- In positive characteristic (i.e $F \cong \mathbb{F}_q((t))$ for some choice of uniformizer $t = \pi_F$), we may consider the module of *continuous* Kähler differentials, whose construction we recall here. First consider the following inverse limit.

$$\hat{\Omega}_{\mathcal{O}_F/\mathbb{F}_q}^1 := \varprojlim_{N \geq 1} \Omega_{(\mathcal{O}_F/\langle \pi_F^N \rangle)/\mathbb{F}_q}^1.$$

Note each term of this inverse limit is given by $\mathcal{O}_F/\langle Nt^{N-1}, t^N \rangle dt$, and thus the limit is a rank one free $\mathcal{O}_F$ -module. By construction we see this to be the initial object in the category of finitely generated $\mathcal{O}_F$ -modules M with an $\mathbb{F}_q$ -linear derivation $\mathcal{O}_F \rightarrow M$. We then take the “generic fiber”

$$\hat{\Omega}_{F/\mathbb{F}_q}^1 := F \otimes_{\mathcal{O}_F} \hat{\Omega}_{\mathcal{O}_F/\mathbb{F}_q}^1,$$

which again by construction is seen to be initial among F -vector spaces V with $\mathbb{F}_q$ -linear derivations $F \rightarrow V$. This is one-dimensional over F , and so any element of it may be written in the form $f dt$ for $f \in F$. Expressing it in its Laurent series expansion $f = \sum a_i t^i$, we may define the **residue** of $\omega = f dt$ (with respect to t) as the coefficient a_{-1} of dt/t . One can show (e.g in Chapter II of [Ser88]) that this value is independent of choice of uniformizer t . Thus, making some choice of nonzero $\omega \in \hat{\Omega}_{F/\mathbb{F}_q}^1$, we may define the nontrivial additive character

$$\psi(x) := \exp\left(\frac{2\pi i}{p} \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\text{Res}(x\omega))\right). \quad (1.2-4)$$

DEFINITION 1.2.2. We will call the characters described in (1.2-1), (1.2-2), (1.2-4) and (1.2-3) as the **standard characters** of their respective local fields.

WARNING. In characteristic zero, the above definition is in some-sense God-given, as the quotient and trace maps involved in the constructions are canonical. In positive characteristic the above definition will always be dependent on some non-canonical *choice* of $\omega \in \hat{\Omega}_{F/\mathbb{F}_q}^1$.

In the function field case, the standard character has the following *global* geometric interpretation: consider X a smooth proper geometrically connected algebraic curve over $\mathbb{F}_q$ with function field K . Let $\Omega_{K/\mathbb{F}_q}^1$ be the one-dimensional K -module of meromorphic 1-forms on X (i.e the stalk at the generic point of the $\mathcal{O}_X$ -module of Kähler differentials $\Omega_{X/\mathbb{F}_q}^1$). Any $\omega \in \Omega_{K/\mathbb{F}_q}^1$ can be written as $f dt$ for some $f \in K$, with some choice of separating transcendence basis $\{t\}$ over $\mathbb{F}_q$. Pick some $v \in \mathcal{P}_K$, corresponding to a point of X , and assume the function $t : X \rightarrow \mathbb{P}_{\mathbb{F}_q}^1$ is unramified and does not map to infinity at this point. Then the coefficient a_{-1} in the expansion of f seen as an element of K_v (which is isomorphic to $\mathbb{F}_q((t - t_v))$) is the **residue** of ω at v , written $\text{Res}_v(\omega)$. We refer the reader to Chapter II of [Ser88] for a more detailed exposition. Later we will see that this formulation is practically necessary for defining an adelic character in a clean way.

We now build the notion of a conductor of a (quasi-)character, which will be an invaluable tool in computations later.

DEFINITION 1.2.3. Let ψ be a nontrivial additive quasi-character on a non-Archimedean local field F . Then the **conductor** of ψ is the ideal $\mathfrak{p}^m$, where

$$m := \min \{r \in \mathbb{Z} : \psi|_{\mathfrak{p}^r} = 1\},$$

i.e the conductor is the largest fractional ideal of $\mathcal{O}_F$ on which ψ is trivial.

Now suppose χ is a multiplicative quasi-character on $F^\times$. Then its **conductor** is $\mathfrak{p}^n$, where

$$n := \min \{r \geq 0 : \chi|_{1+\mathfrak{p}^r} = 1\},$$

i.e the conductor is the largest possible unit congruence subgroup of $\mathcal{O}_F$ on which χ is trivial. By convention, if χ is trivial on $\mathcal{O}_F^\times$ then its conductor is $\mathcal{O}_F$ (i.e $n = 0$).

REMARK (Terminology). The origin of the term "conductor" comes from Dedekind, who denoted by "*Führer*" (whose meaning was closer to "guide", "attendant" or "conductor" in the original German, rather than its modern associations) a nonzero ideal of an order in a number field, which is why classical texts denote conductors using $\mathfrak{f}$. Sometimes the term "conductor" also refers to integers m and n themselves rather than just the ideals. In the case that $m = 0$ (resp. $n = 0$), we will say that ψ (resp. χ) has **trivial conductor**.

THEOREM 1.2.4 (Local-Fields are Self-Dual). Let F be a local field equipped with an absolute value $|\cdot|$, and fix a non-trivial continuous additive character $\psi \in \widehat{F}$. Then the map

$$\begin{aligned} \Psi : F &\mapsto \widehat{F} \\ a &\mapsto (x \mapsto \psi(ax)) \end{aligned}$$

is an isomorphism of topological groups.

Proof. Since left-multiplication gives a continuous automorphism of the additive group of a field, we immediately see the map Ψ is a well-defined group homomorphism. We also see it is injective by the fact that ψ was assumed to be non-trivial, so $\ker(\Psi) = 0$. So it remains to show Ψ is a surjective homeomorphism.

We begin by showing Ψ is a continuous map. For $\lambda, \epsilon \in \mathbb{R}_{>0}$, note sets of the form

$$W(\lambda, \epsilon) := \left\{ \chi \in \widehat{F} : |1 - \chi(x)| \leq \epsilon, \forall x \in \overline{B_\lambda(0)} \subseteq F \right\}$$

form a neighborhood base for the trivial character in the compact-open topology. Fix such an open set for choice of ϵ, λ , and now for continuity it suffices to show there exists some open neighborhood U of $0 \in F$ carried into $W(\lambda, \epsilon)$ via Ψ . However, note by continuity of our original character ψ that there exists some $\delta > 0$ such that

$$x \in B_\delta(0) \implies |1 - \psi(x)| < \epsilon.$$

Then note the open set $U := B_{\delta/\lambda}$ suffices, since for any $y \in U$ and $x \in \overline{B_\lambda(0)}$ we have $|xy| < \delta$, and so $|1 - \Psi(y)(x)| < \epsilon$. Thus Ψ is continuous. Running the construction in reverse, we can also easily verify that Ψ is in fact an open map (thus a homeomorphism) onto its image.

Finally we need to show surjectivity. We claim that the image $H := \Psi(F)$ is first of all complete (as a topological group, not a metric space!). Take any Cauchy sequence $\{\Psi(x_n)\}$ in H , then we claim $\{x_n\}$ must also be Cauchy in F . Suppose not, then there exists an infinite sequence of distinct (and hence arbitrarily large) positive integers $\{n_i\}_i$ and some $\epsilon > 0$ such that we could bound $|x_{n_i} - x_{n_j}| > \epsilon$ for all $i \neq j$. Fixing some $c \in F$ for which

$\psi(c) \neq 1$, which exists by the nontriviality assumption, we may consider closed ball of radius $\lambda := |c|_F / \epsilon$. By construction then, for any $i \neq j$ we would have some $y_{i,j} \in F$ such that $(x_i - x_j)y_{i,j} = c$. However, then we would have

$$\left| 1 - \Psi(x_{n_i} - x_{n_j})(y_{i,j}) \right| = \left| 1 - \psi(x_{n_i} - x_{n_j})(y_{i,j}) \right| = |1 - \psi(c)| \quad \forall i \neq j.$$

But this contradicts the Cauchy property of $\{\Psi(x_n)\}$ in the compact open topology, which says that for all n larger than some fixed N we must have $|1 - \Psi(x_n)(y)| < |1 - \psi(c)|$ for all $y \in \overline{B}_{0,\lambda}$. Thus the sequence $\{x_n\}$ must also be Cauchy.

In particular, completeness of F implies completeness of its image H under Ψ , and thus H is a closed subgroup of $\widehat{F}$. Therefore, both H and $\widehat{F}/H$ are LCA groups, and by Pontryagin duality considerations we can identify

$$\widehat{\widehat{F}/H} \cong \left\{ x \in \widehat{\widehat{F}} (= F) : x(h) = 1 \quad \forall h \in H \right\},$$

where of course by x we mean the "evaluation at x " map. But by nontriviality of our original ψ , there exists some $y \in F$ for which $\psi(y) \neq 1$, so for any $x \in F \setminus \{0\}$ we note that $\psi_{yx^{-1}} \in H$ evaluates to $\psi(y) \neq 1$ on x . It follows that $\widehat{\widehat{F}/H}$ is trivial, from which we deduce $H = \widehat{F}$, giving us surjectivity. $\square$

We now proceed to discuss the theory of characters on the adelic groups, with the key result being that of Pontryagin duality commuting with the restricted product. This result in fact holds for any restricted product, so the following analysis is done in greater generality so we can hit the adeles and ideles with one stone, as it were.

LEMMA 1.2.5. As with the notation of Definition 1.1.1, let $\{G_v\}_v$ be a countable set of locally-compact abelian Hausdorff groups with compact open respective subgroups H_v for almost all v , and let G be their restricted product.

- (a) (Breaking Down) Any quasi-character $\chi \in \text{Hom}_{\text{cont}}(G, \mathbb{C}^\times)$ is trivial on almost all H_v .
- (b) (Building Up) Fix quasi-characters $\chi_v \in \text{Hom}_{\text{cont}}(G_v, \mathbb{C}^\times)$ such that $\chi_v|_{H_v} = 1$ for almost all v . Then $\chi := \prod \chi_v$ lies in $\text{Hom}_{\text{cont}}(G, \mathbb{C}^\times)$.

The same results hold if we replace every instance of "quasi-character" with "character".

Proof. Breaking Down

Since $\mathbb{C}^\times$ has "no small subgroups", we may fix a neighborhood $V \ni 1$ that does not contain any non-trivial subgroup. By considering pre-images we know there must exist some open neighborhood $U = \prod U_v$ of the identity of G such that $\chi(U) \subseteq V$. Let S denote the (finite!) index set such that $U_v \neq H_v$ exactly when $v \in S$. Consider the following subgroup of G :

$$H := \prod_{v \in S} 1 \times \prod_{v \notin S} H_v \subseteq U.$$

Note that $\chi(H) \subseteq U$, but since its image must be a group it must be $\{1\}$, giving us what we want. In particular, for any $x = (x_v)_v \in G$, if we define

$$\chi_v(x_v) := \chi(\dots, 1, 1, x_v, 1, 1, \dots),$$

we deduce for any x that $\chi_v|_{H_v} = 1$ almost everywhere, and $\chi = \prod \chi_v$.

Building Up

By checking definitions one confirms that χ is a well-defined homomorphism $G \rightarrow \mathbb{C}^\times$, so it remains to verify continuity. Let S denote the (finite!) set of indices where $\chi_v|_{H_v} \neq 1$, and let $s := |S|$. For any subset $X \subseteq G$, we can define

$$X^{(s)} := \left\{ \prod_{i=1}^s g_i : g_i \in X \right\},$$

i.e. the set of all elements of G that are formed as products of exactly s elements of X . Note that for any neighborhood $V \ni 1$ in $\mathbb{C}^\times$, there must exist some other neighborhood W such that $W^{(s)} \subseteq V$ (one could achieve this by taking an arbitrarily small ball around 1, for example). By a standard result of character theory, for every $v \in S$ we must have some neighborhood U_v of the identity of G_v such that $\chi(U_v) \subseteq W$. It follows that:

$$\prod_{v \in S} U_v \times \prod_{v \notin S} H_v \subseteq \chi^{-1}(W^{(s)}) \subseteq \chi^{-1}(V),$$

from which it follows that χ is continuous. $\square$

We now have the tools needed to construct the dual group of the restricted product. For each component G_v , by local compactness we have the dual group $\widehat{G_v}$ of continuous characters that satisfies Pontryagin duality. Let J_∞ be the finite index set corresponding to infinite places, then for every $v \notin J_\infty$ we define

$$K(G_v, H_v) := \left\{ \chi_v \in \widehat{G_v} : \chi_v|_{H_v} = 1 \right\},$$

i.e. characters with trivial restriction to H_v . By selecting a neighborhood $V \ni 1$ in S^1 small enough to not contain any non-trivial subgroups of S^1 , we note that $K(G_v, H_v)$ is a basis element of the compact-open topology of $\widehat{G_v}$, and so is open in $\widehat{G_v}$.

Since $\chi_v \in K(G_v, H_v)$ is trivial on H_v , it factors through the quotient G_v/H_v , inducing a topological group isomorphism

$$K(G_v, H_v) \xrightarrow{\sim} \widehat{G_v/H_v}.$$

By openness of $H_v \subseteq G_v$, we get that the quotient G_v/H_v is a discrete group, and standard character theory, the dual $\widehat{G_v/H_v} \cong K(G_v, H_v)$ must be compact. As such, with respect to the compact open subgroups $K(G_v, H_v)$, it makes sense to define the restricted product of the dual groups $\widehat{G_v}$.

THEOREM 1.2.6 (Restricted Product Commutes with Dual Groups). With the notation above, we have an isomorphism of topological groups

$$\widehat{G} \xrightarrow{\sim} \prod \widehat{G_v}.$$

Proof. Consider the map

$$\begin{aligned} \phi : \prod \widehat{G_v} &\rightarrow \widehat{G} \\ (\chi_v)_v &\mapsto \prod_v \chi_v. \end{aligned}$$

By the construction of the restricted product and Lemma 1.2.5, this is a well-defined group isomorphism. It remains to show ϕ is a homeomorphism.

By translation invariance of open sets in topological groups it suffices to prove bicontinuity at the identity in the domain and codomain. From part (b) of Proposition 1.1.3, we recall any compact neighborhood of the identity

$K \subseteq G$ is of the form $\prod K_v$ for compact K_v that are equal to H_v almost everywhere. For U an open neighborhood of $1 \in S^1$, consider the open neighborhood of the identity

$$W(K, U) := \left\{ \chi \in \widehat{G} : \chi(K) \subseteq U \right\},$$

i.e a basis element of the compact-open topology on $\widehat{G}$. We want to show any such $W(K, U)$ is open. For any $\chi \in W(K, U)$, let S be the (finite!) set of indices for which χ is non-trivial on K_v , and let $s = |S|$. As in the Lemma lemma: characters on restricted products, we can select some neighborhood $V \ni 1$ in $\mathbb{C}^\times$ such that $V^{(s)} \subseteq U$, and consider the following open neighborhood of the identity in $\prod \widehat{G}_v$:

$$X := \prod_{v \in S} W(K_v, V) \times \prod_{v \notin S} K(G_v, H_v) \subseteq_{\text{open}} \prod \widehat{G}_v.$$

We note that $\phi^{-1}(\chi) \in X$ by construction, so in particular we have $\phi(X) \subseteq W(K, U)$, and so ϕ is continuous. For continuity in the other direction, let $X = \prod W(K_v, U)$ be an open neighborhood of the identity in $\prod \widehat{G}_v$. Then, note

$$\widehat{G} \supseteq W(\prod K_v, U) \supseteq \phi(X),$$

proving continuity in the other direction. □

The above result tells us the following neat isomorphism for the dual adele and idele groups, which will allow us to work component-wise without worry.

COROLLARY 1.2.7. For a global field K ,

$$\widehat{\mathbb{A}_K} \cong \prod \widehat{K_v}, \quad \widehat{\mathbb{I}_K} \cong \prod \widehat{K_v^\times},$$

where the restricted products are taken with respect to the various (compact!) $\widehat{K_v/\mathcal{O}_v}$ and $\widehat{K_v^\times/\mathcal{O}_v^\times}$ respectively.

The key step to showing self duality of the adeles (which is ultimately what will put our adelic Fourier theory on a sound basis) will be the construction of *some* nontrivial additive character on $\mathbb{A}_K$ with the property that it is trivial on the canonically embedded $K \hookrightarrow \mathbb{A}_K$. This property will be key to deducing results like the adelic Poisson summation formula in §3.1, which will be crucial to showing the functional equation of the global zeta function in §3.2 – the crux of Tate’s thesis!

PROPOSITION 1.2.8. For any global field K , there exists a non-trivial character ψ on $\mathbb{A}_K$ that is trivial on the image of K under the diagonal embedding. We call these **standard** adelic characters.

Proof. When K is a number field, on each component we may choose the character defined in Definition 1.2.2; since this is given by $\psi_v := \psi_p \circ \text{Tr}_{K_v/\mathbb{Q}_p}$ and ψ_p has conductor $\mathbb{Z}_p$ by construction, it is clear that ψ_v has conductor $\text{Tr}^{-1}(\mathbb{Z}_p) = \mathfrak{D}_v^{-1}$. This is trivial – i.e equal to $\mathcal{O}_v$ – at all the (cofinitely many!) $v \in \mathscr{P}_K$ unramified over $\mathbb{Q}$, so the global character

$$\begin{aligned} \psi_K : \mathbb{A}_K &\rightarrow S^1 \\ (x_v) &\mapsto \prod_v \psi_v(x_v). \end{aligned}$$

is in fact well-defined. This character is non-trivial, since ψ_v are all non-trivial characters, so we can see that the $\psi(\dots, 1, x_v, 1, \dots) \neq 1$ for any x_v such that $\psi_v(x_v) \neq 1$.

To verify triviality on K it suffices to consider the case $K = \mathbb{Q}$ by the identity $\psi_K = \psi_{\mathbb{Q}} \circ \text{Tr } \mathbb{A}_K / \mathbb{A}_{\mathbb{Q}}$ (which is easily verified from our construction in (1.2-3)). For any $x \in \mathbb{Q}$, recall we have a unique factorization

$$x = b + \sum_{p \text{ prime}} \frac{a_p}{p^{\text{ord}_p(x)}}, \quad a_p, b \in \mathbb{Z},$$

where a_p is nonzero for only finitely many primes. Recalling the expression for our ψ_v defined in (1.2-1) and (1.2-2), we have:

$$\begin{aligned} \psi_{\mathbb{Q}}(x) &= \prod_{v \in \mathcal{P}_{\mathbb{Q}}} \psi_v(x) \\ &= e^{-2\pi i x} \prod_{p \text{ prime}} e^{2\pi i a_p / p^{\text{ord}_p(x)}} \\ &= e^{-2\pi i b}, \end{aligned}$$

which we deduce is equal to 1 since $b \in \mathbb{Z}$. Thus ψ is trivial on $\mathbb{Q}$, as desired.

When K is a function field, we recall from (1.2-4) that the standard character on each K_v is given by some choice of $\omega_v \in \widehat{\Omega}_{K_v/\kappa_v}^1$, but how to make such a choice on every place? We use the *global* meromorphic 1-forms discussed in the paragraph following Definition 1.2.2. Fixing some global nonzero $\omega \in \Omega_{K/\mathbb{F}_q}^1$, for any $v \in \mathcal{P}_K$, if κ_v denotes the residue field at v we have the residue map

$$\text{Res}_v : K_v \otimes_K \Omega_{K/\mathbb{F}_q}^1 (= \widehat{\Omega}_{K_v/\mathbb{F}_q}^1) \rightarrow \kappa_v.$$

We may then define

$$\psi_v(x_v) := \exp \left(\frac{2\pi i}{p} \text{Tr}_{\kappa_v/\mathbb{F}_p}(\text{Res}_v(x_v \omega)) \right)$$

To see that this is well-defined we note at each v this has conductor precisely $\mathfrak{p}_v^{-d_v}$, where d_v is the order of vanishing of ω at v . More precisely, it is the largest integer so that $\omega \in \mathfrak{p}_v^{d_v} \Omega_{X,v}^1$. But of course ω is regular and nonvanishing at cofinitely many points, so ψ_v has trivial conductor at all such places.

For triviality on K we make use of the classical result known as *residue formula* (see Sections II-12 and II-13 in [Ser88]), which says that the sum of the trace (to $\mathbb{F}_q$) of the residues taken over all places of K for any $\omega \in \Omega_{K/\mathbb{F}_q}^1$ is well defined and equal to 0. The key point here is that $x\omega$ is yet another meromorphic 1-form for any $x \in K$. We thus get:

$$\begin{aligned} \psi_K(x) &= \prod_{v \in \mathcal{P}_K} \psi_v(x) = \prod_v \exp \left(\frac{2\pi i}{p} \text{Tr}_{\kappa_v/\mathbb{F}_p}(\text{Res}_v(x\omega)) \right) \\ &= \exp \left(\frac{2\pi i}{p} \text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} \left(\sum \text{Tr}_{\kappa_v/\mathbb{F}_q}(\text{Res}_v(x\omega)) \right) \right) \\ &= e^0 = 1, \end{aligned}$$

as desired. □

COROLLARY 1.2.9 (Adeles are self-dual). Let K be a global field, and ψ a standard non-trivial character of $\mathbb{A}_K$,

with ψ_v being nontrivial for all K_v and having conductor $\mathcal{O}_v$ for cofinitely many v . Then the map

$$\begin{aligned}\Psi : \mathbb{A}_K &\rightarrow \widehat{\mathbb{A}_K} \\ a &\mapsto (\psi_a : x \mapsto \psi(ax)),\end{aligned}$$

is an isomorphism of topological groups.

Proof. The proof of Proposition 1.2.8 constructed such a character ψ on $\mathbb{A}_K$. By nontriviality of ψ_v , we know that $\Psi_v : K_v \xrightarrow{\sim} \widehat{K_v}$ is an isomorphism by Theorem 1.2.4. Moreover, since by construction the local character ψ_v has conductor $\mathcal{O}_v$ at cofinitely many places, we have an induced identification of $\mathcal{O}_v$ with $\widehat{K_v/\mathcal{O}_v}$ at all such places. Then Corollary 1.2.7 tells us the dual of the restricted product is the restricted product of the duals of each component, giving us our desired homeomorphism. $\square$

PROPOSITION 1.2.10. The map

$$\begin{aligned}\Psi' : K &\rightarrow \widehat{\mathbb{A}_K/K} \\ a &\mapsto (\psi_a : x \mapsto \psi(ax))\end{aligned}$$

is an isomorphism of discrete topological groups. In particular, if we fix some standard ψ on $\mathbb{A}_K$ with the properties stated in Corollary 1.2.9, then ψ_a is standard on K if and only if $a \in K^\times \subseteq \mathbb{A}_K$.

Proof. As we will see later in Theorem 1.3.1, $K \subseteq \mathbb{A}_K$ is a discrete closed subgroup and the quotient $\mathbb{A}_K/K$ is compact. By a standard result of Pontryagin duality, the dual $\widehat{\mathbb{A}_K/K}$ must be discrete, and in particular homeomorphically embed as a discrete closed subgroup of $\widehat{\mathbb{A}_K}$. By Corollary 1.2.9 we've seen that $\widehat{\mathbb{A}_K} \cong \mathbb{A}_K$, so we have that $\widehat{\mathbb{A}_K/K}/K$ is a closed, discrete K -subvector space of the compact $\mathbb{A}_K/K$, from which we deduce the cardinality of $\widehat{\mathbb{A}_K/K}/K$ must be finite. However, any global field has infinite cardinality, so we must have $\widehat{\mathbb{A}_K/K} = K$, completing our proof. $\square$

REMARK. There is a more canonical way to see this duality when K is of positive characteristic, though it requires some more algebraic geometry to formulate and appreciate appropriately. In the language of §3.1, we may identify K with the function field of a smooth projective curve X . For any divisor D on X , one can define a line bundle $\mathcal{L}(D)$, whose nonzero sections over any open $U \subseteq X$ consist of all the functions $f \in K^\times$ satisfying $\text{div}_U(f) \geq -D \cap U$. Then there is a *canonical* isomorphism

$$H^1(X, \mathcal{L}(D)) \xrightarrow{\sim} \mathbb{A}_K/(\mathbb{A}_K(D) + K),$$

where $\mathbb{A}_K(D)$ is all the adeles $(x_v)_v$ such that $\text{ord}_v(x_v) \geq -\text{ord}_v(D)$ (by convention $\text{ord}_v(0) = \infty$). The dual of the right side of the above is (canonically, via Serre duality) isomorphic to $H^0(X, \Omega_X^1 \otimes \mathcal{L}(D)^*) = \Omega_X^1(D)$ (i.e global 1-forms with poles at worst D). Taking a directed union over all D gives

$$\Omega_{K/\mathbb{F}_q}^1 \xrightarrow{\sim} \widehat{\mathbb{A}_K/K}.$$

The left side of the above is clearly 1-dimensional over K , which gives us our desired result. Chapter II of [Ser88] gives a more detailed (albeit with some outdated notation) account of this.

The above results motivate the following definition.

DEFINITION 1.2.11. For an LCA group equipped with self-duality, a Haar measure dx is **self-dual** if it is equal to its own dual measure (i.e the measure with respect to which the Fourier inversion formula holds).

Our goal next will be to find an explicit measure dx_v on K_v that is self-dual (with respect to the isomorphism $K_v \xrightarrow{\sim} \widehat{K_v}$ induced by some *choice* of non-trivial $\psi_v \in \widehat{K_v}$) for each $v \in \mathcal{P}_K$. We begin by fleshing out the theory of measures on the general restricted product; specifically, we want a Haar measure that is a "restricted product" of Haar measures on each of the components. This will require a bit of care, since the restricted product is not actually a product space.

PROPOSITION 1.2.12. For all but finitely many v , let dg_v be a (left) Haar measure on G_v normalized so that $\text{Vol}(H_v, dg_v) = 1$. Then there is a unique Haar measure dg on G such that the restriction dg_S of dg to any G_S is exactly the product measure for all finite $S \supseteq J_\infty$.

Proof. Choose such a set S satisfying the conditions of finiteness and containing J_∞ , and define dg_S as the product of the measures dg_v . Note the measure of the compact group $\prod_{v \notin S} H_v$ is indeed finite, by how we've normalized things. It is a verification of axioms to deduce the rest that dg_S is indeed a Haar measure on G_S .

Suppose we have another finite set of indices $T \supseteq S$. Then, by construction, it follows that dg_S coincides with the restriction of dg_T to the open subgroup $G_S \subseteq G_T$.

Since G is locally compact, we know it has a (unique up to scaling!) Haar measure, which restricts to a Haar measure on any (clopen!) G_S . Fixing any S , we can scale our Haar measure on G so that when it restricts to G_S it exactly coincides with dg_S . Clearly this construction was independent of choice of S . $\square$

The above result motivates the following.

DEFINITION 1.2.13. Given G the restricted product of LCA groups G_v , we define the **restricted product measure**

$$dg = \prod_v dg_v \tag{1.2-5}$$

as the unique Haar measure induced by Proposition 1.2.12.

Thus to construct a (self-dual) Haar measure on $\mathbb{A}_K$ it suffices to define a (self-dual) Haar measure on each local component K_v that gives measure 1 to $\mathcal{O}_v$ for almost all v .

Since a Haar measure is unique up to scaling, it suffices to specify the finite (positive) volume of any pre-compact set with nonempty interior to determine the measure.

- For $F = \mathbb{R}$, let dx be the standard Lebesgue measure.
- For $F = \mathbb{C}$, let dx be twice the standard Lebesgue measure, i.e. $|dz \wedge d\bar{z}|$.
- For F non-Archimedean, we set $\text{Vol}(\mathcal{O}_F, dx) = (N\mathfrak{D}_F)^{-1/2}$, where we define $\mathfrak{D}_F$ to be the fractional ideal inverse to the conductor of the standard characters defined in Definition 1.2.2. When F has mixed characteristic (i.e arises as a completion of a finite separable extension of $\mathbb{Q}_p$) this is, as expected, the classical different of $F/\mathbb{Q}_p$. In either case, the volume is equal to $q^{-d/2}$, where q is the order of the residue field and $d = \text{ord}_F(\mathfrak{D}_F)$.

PROPOSITION 1.2.14. The measures defined above are self-dual with respect to the standard characters defined in 1.2.2.

Proof. The results for when $F = \mathbb{R}$ or $\mathbb{C}$ are standard in the theory of classical Fourier analysis, so we omit them here. When F is non-Archimedean, recall that the Fourier inversion theorem tells us that it suffices to check for any nonzero Schwartz-Bruhat function. To that end, let f be the characteristic function of $\mathcal{O}_F$. We thus get

$$\widehat{f}(y) = \int_F f(x) \overline{\psi(xy)} dx = \int_{\mathcal{O}_F} \overline{\psi(xy)} dx.$$

We are (slightly) abusing notation here since $\widehat{f}$ should be a function on $\widehat{F}$, but Theorem 1.2.4 allows us to do this with no worries.

We note that the different is the the inverse of the largest fractional ideal of F on which ψ is trivial; this is by construction in characteristic 0 and by definition in positive characteristic. Thus, for all $x \in \mathcal{O}_F$, we have $\psi(xy) = 1$ if and only if $y \in \mathfrak{D}_F^{-1}$. This gives us

$$\widehat{f}(y) = \begin{cases} \text{Vol}(\mathcal{O}_F, dx) = (N\mathfrak{D}_F)^{-1/2}, & y \in \mathfrak{D}_F^{-1} \\ 0, & \text{otherwise,} \end{cases}$$

where the second equality follows from the fact that we are integrating a nontrivial character over a compact group. Taking the Fourier transform again we get:

$$\widehat{\widehat{f}}(x) = \int_F \widehat{f}(y) \overline{\psi(xy)} dy = \int_{\mathfrak{D}_F^{-1}} (N\mathfrak{D}_F)^{-1/2} \overline{\psi(xy)} dy,$$

where again we slightly abuse notation here using the isomorphism of F with its dual. By similar logic to the previous step, for any $x \in F$, we note that $\psi(xy)$ restricted to the codifferent is trivial if and only if $x \in \mathcal{O}_F$. Thus, we have:

$$\widehat{\widehat{f}}(x) = \begin{cases} (N\mathfrak{D}_F)^{-1/2} \text{Vol}(\mathfrak{D}_F^{-1}, dx), & x \in \mathcal{O}_F \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we see that

$$\text{Vol}(\mathfrak{D}_F^{-1}, dx) = N\mathfrak{D}_F \text{Vol}(\mathcal{O}_F, dx) = (N\mathfrak{D}_F)^{1/2},$$

and thus $\widehat{\widehat{f}}$ is the characteristic function of $\mathcal{O}_F$. Thus $\widehat{\widehat{f}} = f$, and in particular $\widehat{\widehat{f}}(x) = f(-x)$ since $\mathcal{O}_F$ is an additively closed subgroup, so the relevant Fourier inversion formula holds, proving self-duality of the measure. $\square$

We are now ready to define the standard measure on the adeles.

DEFINITION 1.2.15. For every place $v \in \mathcal{P}_K$, let dx_v denote the self-dual Haar measure of K_v with respect to the standard characters defined in 1.2.2. Then the **Tamagawa measure** on $\mathbb{A}_K$ is given by

$$dx := \prod_v dx_v.$$

SANITY CHECK. It's good to verify that the above is actually well defined. By Proposition 1.2.12, it suffices to show that the self-dual Haar measures satisfy $\text{Vol}(\mathcal{O}_v, dx_v) = 1$ for almost all finite places v . This is true by the fact the different is trivial (i.e equal to $\mathcal{O}_v$) at all the unramified – hence cofinitely many – places.

If we impose the counting measure on K , that combined with the Tamagawa measure on $\mathbb{A}_K$ induce a unique Haar measure on the quotient $\mathbb{A}_K/K$ compatible with the exact sequence of topological groups

$$0 \rightarrow K \rightarrow \mathbb{A}_K \rightarrow \mathbb{A}_K/K \rightarrow 0.$$

THEOREM 1.2.16. The volume of $\mathbb{A}_K/K$ (with respect to the uniquely defined measure discussed above) is 1.

Proof. One can perform an explicit computation with fundamental domains to compute this, but this can actually be deduced in a much cleaner, computation-free fashion as a consequence of the adelic Riemann-Roch theorem. Thus, we mention this result here for narrative flow but will deduce it in Corollary 3.1.10. In particular, it implies the Tamagawa measure is canonical even in the function field case! $\square$

REMARK. The above theorem is the starting point of some rather deep theory of a conjecture of Weil. Hand-waving a lot of details, for any global field K , if we normalize measures as done above so the quotient $\mathbb{A}_K/K$ has volume 1, then the volume of the certain Haar measure of $G(K) \backslash G(\mathbb{A}_K)$ is 1, for any simply connected (as an algebraic group, NOT a topological space!) semisimple algebraic group G . This conjecture was proven in the number field case in the 1980s by Langlands, Lai and Kottwitz, and in the global function field case by Gaitsgory and Lurie in 2011.

§1.3 Adelic Approximations and Quotients

In this subsection we outline various compactness and discreteness results of subsets and quotients of the adeles and ideles that will become invaluable to us when performing adelic analysis in Chapter II.

THEOREM 1.3.1. Any global field K is a discrete, closed, cocompact subgroup of its adèle ring, i.e. $\mathbb{A}_K/K$ is compact.

Proof. Let $K_0 = \mathbb{Q}$ in characteristic 0 or $\mathbb{F}_p(t)$ for some *choice* of separating transcendence bases $\{t\}$ in positive characteristic, and let $n := [K : K_0]$. Then by 1.1.9, we have the following commutative diagram:

$$\begin{array}{ccc} \prod_{j=1}^n \mathbb{A}_{K_0} & \longrightarrow & \mathbb{A}_K \\ \uparrow & & \uparrow \\ \prod_{j=1}^n K_0 & \longrightarrow & K \end{array}$$

where the horizontal maps are topological isomorphisms. Thus $\mathbb{A}_K/K$ is compact if and only if $(\mathbb{A}_{K_0}/K_0)^n$ is compact, which is true if and only if $\mathbb{A}_{K_0}/K_0$ is compact.

Thus it suffices to consider $K = K_0$, in which case we have exactly one infinite place (which in the case of positive characteristic corresponds to the "point at infinity" on $\mathbb{P}_{\mathbb{F}_p}^1$). Define the closed neighborhood C of 0 in $\mathbb{A}_K$ by

$$C := \left\{ x \in \mathbb{A}_K : |x_\infty|_\infty \leq \frac{1}{2} \text{ and } |x_v|_v \leq 1, \forall v \neq \infty \right\}$$

Note the condition on the finite places v simply amounts to saying that $x_v \in \mathcal{O}_v$. If we show that $C \cap K = \{0\}$ and $\mathbb{A}_K = C + K$, we will be done, since C is visibly compact and isomorphic to $\mathbb{A}_K/K$.

If $x \in K \cap C$, then we must have $x \in \mathcal{O}_K$ (where in positive characteristic we may define $\mathcal{O}_K := \mathbb{F}_p[t]$, with the same choice of t as discussed before), since $x \in \mathcal{O}_v$ for every $v \nmid \infty$. If $K = \mathbb{Q}$ then this condition ($x \in \mathbb{Z}$) combined with the condition that $|x_\infty|_\infty \leq 1/2$ forces $x = 0$. If $K = \mathbb{F}_p(t)$, then $|x|_\infty = p^{-\text{ord}_{1/t}(x)}$, which is ≥ 1 for anything nonzero in $\mathcal{O}_K = \mathbb{F}_p[t]$. Thus $C \cap K = \{0\}$.

It remains to show, with $K = K_0$, that any $x = (x_v)_v \in \mathbb{A}_K$ can be written as $y + c$, for y a principal adele and $c \in C$. At each finite place v we have the isomorphism

$$K_v/\mathcal{O}_v \xrightarrow{\sim} K/(\mathcal{O}_{K, \mathfrak{p}_v}),$$

with $\mathcal{O}_K$ a PID. Thus, for each finite v we may pick a uniformizer π_v for $\mathcal{O}_v$ such that $\pi_v \in \mathcal{O}_K$ and is a unit at all other finite places. Using this, at all the (finitely many!) places where $x_v \notin \mathcal{O}_v$ we may subtract some $\mathcal{O}_K$ -unit multiplied by sufficient power of π_v (i.e. some element of K) from x_v so that the difference lies in $\mathcal{O}_v$. Moreover by construction this does not affect the valuation of any other element at any other place. Since there are only finitely many such subtractions to make, we can subtract a well-defined element $y_v \in K$ from $x = (x_v)_v$ so that $x_v \in \mathcal{O}_v$ at all $v < \infty$. At the infinite place we may simply pick some $y_\infty \in \mathbb{Z}$ so that $|x_\infty - y_\infty|_\infty \leq 1/2$ (i.e. just pick the nearest integer) or $y_\infty \in \mathbb{F}_p[t]$ so that $|x_\infty - y_\infty|_\infty = p^{-1}$ (i.e. just pick a polynomial equal to the "polar part" of $x_\infty \in \mathbb{F}_p[1/t]$). By construction then, if we let S denote the finite set of places where $x_v \notin \mathcal{O}_v$, we have:

$$y := y_\infty + \sum_{v \in S} y_v \in K, \quad x - y \in C,$$

as desired. □

COROLLARY 1.3.2. The image of the diagonal embedding of $K^\times$ is discrete in $\mathbb{I}_K$.

Proof. Via the map $K^\times \hookrightarrow \mathbb{I}_K \subseteq \mathbb{A}_K \times \mathbb{A}_K$ given by $u \mapsto (u, u^{-1})$, (recalling the ideles are topologically embedded in this way) restricting to the first coordinate immediately gives us our result. □

REMARK. The discreteness of the image of the embedding is a more delicate fact than one might think on first inspection, in the sense that if we omit *any* place, the image of the diagonal embedding is also dense! We will see this when discussing strong approximation (Corollary 1.3.13) later on.

EXAMPLE 1.3.3 (Volume of Quotient in non-Tamagawa Measure). For various purposes one may not always want to use the Tamagawa measure on $\mathbb{A}_K$. Here we delve into an example of the types of explicit computations that can arise from using these alternate measures.

Let K be a number field, and let dx be the measure induced on $\mathbb{A}_K$ by setting $\text{Vol}(\mathcal{O}_v, dx_v) = 1$ for all finite places v , and otherwise dx_v is the self-dual measure as defined after Definition 1.2.13. Then:

$$\text{Vol}(\mathbb{A}_K/K, dx) = |\text{disc}(\mathcal{O}_K)|^{1/2}.$$

WARNING (Abuse of Notation). The reader will likely have noticed the measure dx is not actually a measure of the quotient group. What we technically mean is the Haar measure on the quotient induced – via "coset Fubini" – by the counting measure on (the discrete normal subgroup!) K .

Proof Sketch. An examination of the proof of Theorem 1.3.1 shows that the set

$$C := [-1/2, 1/2] \times \prod_{p \text{ prime}} \mathbb{Z}_p$$

is a fundamental domain for the discrete subgroup $\mathbb{Q} \subseteq \mathbb{A}_{\mathbb{Q}}$, so $\text{Vol}(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}, dx) = \text{Vol}(C, dx)$. We immediately see that C has measure $1 = \text{disc}(\mathbb{Z})$.

To generalize the above construction, for $K/\mathbb{Q}$ a finite extension of degree n we may use the isomorphism

$$\eta : \mathbb{R}^n \xrightarrow{\sim} K \otimes_{\mathbb{Q}} \mathbb{R} \left(\cong \prod_{v \in \mathcal{P}_{K, \infty}} K_v \right)$$

via the $\mathbb{Q}$ -basis $\{\xi_1, \dots, \xi_n\}$ of K coming from a $\mathbb{Z}$ -basis of $\mathcal{O}_K$. It is not too hard to show via similar methods that

$$C := \eta([-1/2, 1/2]^n) \times \prod_{v \notin \mathcal{P}_{K, \infty}} \mathcal{O}_v$$

is a fundamental set for K in $\mathbb{A}_K$. If dx_{∞} denotes the product measure over the infinite places of K , then the proof comes down to showing

$$\text{Vol}(\eta([0, 1]^n), dx_{\infty}) = |\text{disc}(\mathcal{O}_K)|^{1/2}.$$

The key trick is to consider the maps

$$\sigma_h : K \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow \mathbb{C}, \quad h \in [1, n]$$

that arise as ($\mathbb{R}$ -linear) extensions of the canonical n distinct embeddings $K \hookrightarrow \mathbb{C}$. The problem then reduces to computing the pushforward measure $\prod_{h \in [n]} d\sigma_h(\theta(u))$, where here θ denotes the inverse of η . One can check this measure to be $|\det(N)| \prod_{h \in [n]} dx_h$, where N is the matrix given by $[(\sigma_h(\xi_i))_i]$, i.e rows are given by the images of $u \in \mathbb{R}^n$ under composition by η followed by σ_h . One can also readily check the determinant of this matrix is the square of the discriminant of K . We refer the reader to Chapter 5 Section 4 of [Wei74] for a more detailed exposition. $\square$

Since we have discrete embeddings $K \hookrightarrow \mathbb{A}_K$ and $K^{\times} \hookrightarrow \mathbb{I}_K$, one might also hope for a similar cocompactness analogue to Theorem 1.3.1; i.e that $K^{\times}$ is also cocompact in $\mathbb{I}_K$. However, this is not the case. To analyze this phenomenon, we first standardize the absolute value functions we're using.

DEFINITION 1.3.4. For K a global field, the **normalized absolute value** on its completion K_v at a place v is defined as follows:

- (a) For $K_v = \mathbb{R}$ we set $|\cdot|_v$ to be the usual absolute value function.
- (b) For $K_v = \mathbb{C}$ we set $|z|_v := z\bar{z}$, which is the square of the usual absolute value function on $\mathbb{C}$.
- (c) For K_v non-Archimedean with uniformizer π and residue field of order q , we set $|\pi|_v := q^{-1}$, which extends uniquely to K_v .

At first glance these normalized absolute values may seem arbitrary and even a bit odd - why set it to be the square of the standard absolute value on $\mathbb{C}$, for example? The answer is these absolute values are precisely the

ones for which $|x|_v$ is the modulus of the automorphism $y \mapsto xy$. That is to say,

$$|x|_v = \frac{\mu(x\mathcal{O}_v)}{\mu(\mathcal{O}_v)},$$

for μ a Haar measure on K_v . Note we may replace $\mathcal{O}_v$ in the above expression with any measurable set of finite positive measure. We leave the verification of these facts to the reader; they are not hard to check.

DEFINITION 1.3.5. Let K be a global field, and let $|\cdot|_v$ denote the normalized absolute value on the completion K_v . Then the **idelic norm** (sometimes called the **idelic absolute value**) is defined by:

$$\begin{aligned} |\cdot|_{\mathbb{I}_K} : \mathbb{I}_K &\rightarrow \mathbb{R}_{>0} \\ x = (x_v)_v &\mapsto \prod_v |x_v|_v \end{aligned}$$

SANITY CHECK. Note that $|\cdot|_{\mathbb{I}_K}$ does indeed converge, since by definition any $x = (x_v) \in \mathbb{I}_K$ is integral (i.e has local norm 1) for cofinitely many v . Moreover, this norm is visibly continuous, as a "product" of continuous norms.

OBSERVATION. The idelic norm of x is, as one might expect, the modulus of the automorphism that is multiplication by x . We leave this fact to be checked by the reader.

THEOREM 1.3.6 (Product Formula). Let K be a global field. Then for every $x \in K^\times$ we have $|x|_{\mathbb{I}_K} = 1$

Proof. Let E/K be finite separable. Then

$$\begin{aligned} |x|_{\mathbb{I}_E} &= \prod_{u \in \mathcal{P}_K} \prod_{v \in \mathcal{P}_E, v|u} |x|_v \\ &= \prod_{u \in \mathcal{P}_K} \prod_{v \in \mathcal{P}_E, v|u} |N_{E_v/K_u}(x)|_u \\ &= \prod_{u \in \mathcal{P}_K} |N_{E/K}(x)|_u, \end{aligned}$$

and since the norm carries $E^\times$ into $K^\times$ showing the result for K proves it for E . In particular, it suffices to prove this for $\mathbb{Q}$ or $\mathbb{F}_q(t)$. Furthermore, multiplicativity of the idelic norm allows us to verify the result for just the irreducibles of $\mathcal{O}_K$.

- For $K = \mathbb{Q}$, since $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$ we can check it for positive primes p and $\mathbb{Z}^\times = \{\pm 1\}$. By definition $|p|_q = 1$ for all primes $q \neq p$. Thus we only have to multiply the p -adic and infinite place, giving us:

$$|p|_{\mathbb{I}_K} = |p|_\infty \cdot |p|_p = p \cdot \frac{1}{p} = 1.$$

The case of $\{\pm 1\}$ is clear.

- For $K = \mathbb{F}_q(t)$, we only have to consider irreducible polynomials $p(t) \in \mathbb{F}_q[t]$, and $\mathbb{F}_q[t]^\times = \mathbb{F}_q^\times$. Again this has trivial absolute value at all but two places. This gives us:

$$|p(t)|_{\mathbb{I}_K} = |p(t)|_\infty \cdot |p(t)|_{p(t)} = q^{\deg p} \cdot (|\mathbb{F}_q[t]/\langle p(t) \rangle|)^{-1} = 1.$$

The case of $\mathbb{F}_q^\times$.

This completes our proof. □

REMARK. Some texts and notes (including [RV98] and the "Adele ring" Wikipedia page) refer to 1.3.6 as the *Artin product formula*. A less trivial result, also due to Artin, is the converse: that any field K for which the "product formula" holds must be a global field. For a more precise statement with proof, we direct the reader to the 1945 paper of Artin and Whaples [AW45].

EXAMPLE 1.3.7 (Decomposition of $\mathbb{I}_{\mathbb{Q}}$). We claim we can topologically describe the idele group over $\mathbb{Q}$ as follows:

$$\mathbb{I}_{\mathbb{Q}} = \mathbb{Q}^{\times} \times \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^{\times},$$

where $\widehat{\mathbb{Z}} := \varprojlim_{n \in \mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, with the inverse limit taken via the natural maps induced by divisibility. The notation is unfortunate here; the hat here signifies completion, not Pontryagin dual.

To see this, consider the map

$$\begin{aligned} \phi : \mathbb{I}_{\mathbb{Q}} &\rightarrow \mathbb{Q}^{\times} \\ x = (x_v)_v &\mapsto \text{sgn}(x_{\infty}) \prod_p |x_p|_p^{-1}, \end{aligned}$$

which has open kernel. Recalling our normalized absolute values defined in 1.3.4, it is immediate from its construction that $\phi(x) = x$ for any $x \in \mathbb{Q}^{\times} \subseteq \mathbb{I}_{\mathbb{Q}}$. Thus, $\mathbb{I}_{\mathbb{Q}} = \mathbb{Q}^{\times} \times \ker(\phi)$ topologically. It is also immediate by construction that $\ker(\phi) = \mathbb{R}_{\geq 0} \times \prod_p \mathbb{Z}_p^{\times}$, and the identity $\widehat{\mathbb{Z}}^{\times} \cong \prod_p \mathbb{Z}_p^{\times}$ completes our proof.

OBSERVATION. The product formula allows us to generalize the above example to see why $\mathbb{I}_K/K^{\times}$ is *never* compact.

When K is a number field, consider the idele

$$x := (t, 1, 1, \dots),$$

which is 1 everywhere except a single infinite place. Note then that $|x|_{\mathbb{I}_K} = t$ or t^2 , depending on whether the place is real or complex. Regardless, this observation tells us that $\mathbb{I}_K$ continuously surjects onto $\mathbb{R}_{>0}$ (a non-compact image!) under the continuous idelic norm map.

When K has positive characteristic with field of constants $\mathbb{F}_q$, the residue field at any completion is a finite extension of $\mathbb{F}_q$, so the norm of any idele must be a power of q . The product formula then tells us that the image of the idelic norm must be some nontrivial and hence finite-index subgroup of $q^{\mathbb{Z}}$, thus noncompact (as an infinite discrete group).

Example 1.3.7 gave an explicit description of the non-compactness of $\mathbb{I}_K/K^{\times}$, but it fortunately also provides a direction in which we may amend this.

DEFINITION 1.3.8. For K a global field and the idelic norm $|\cdot|_{\mathbb{I}_K} : \mathbb{I}_K \rightarrow \mathbb{R}_{>0}$, define the **ideles of norm 1** to be:

$$\mathbb{I}_K^1 := \ker(|\cdot|_{\mathbb{I}_K})$$

and the **norm-1 idele class group** to be the quotient:

$$C_K^1 := \mathbb{I}_K^1 / K^\times$$

We have the short-exact sequence

$$1 \rightarrow C_K^1 \rightarrow C_K \rightarrow |\mathbb{I}_K|_{\mathbb{I}_K} \rightarrow 1$$

which is in fact split-exact in the number field case, by our previous reasoning for non-compactness of C_K . We should guess that this C_K^1 should be the compact quotient we're looking for. Before getting into that, we make an interesting observation about the norm-1 ideles that will come in useful later.

LEMMA 1.3.9. The subspace topologies of $\mathbb{I}_K^1$ taken with respect to $\mathbb{I}_K$ or $\mathbb{A}_K$ are the same.

Proof. Suppose we give $\mathbb{I}_K^1$ the idelic subspace topology by default; it then suffices to show that the injection $\phi : \mathbb{I}_K^1 \hookrightarrow \mathbb{A}_K$ is a homeomorphism onto its image. We have seen in Example 1.1.7 that the subspace adelic topology is coarser than the idelic topology, so ϕ is automatically continuous.

To show it is an open map onto its image, it suffices to show any basic open set of $\mathbb{I}_K^1$ contains a subset open in the adelic subspace topology. To this end, take some basic open neighborhood of the multiplicative identity $1 \in \mathbb{I}_K^1$ in the idelic subspace topology. This has the form:

$$U = \left(\prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v^\times \right) \cap \mathbb{I}_K^1$$

where each U_v is some open ball around $1 \in K_v^\times$, and S is some finite index set containing the infinite places. Now consider the set

$$W := \prod_{v \in S} W_v \times \prod_{v \notin S} \mathcal{O}_v,$$

where the index set S is the same as in the description of U , each $W_v \subseteq U_v$, shrunk appropriately so that for every $\gamma \in W$ satisfies the (strict!) inequality

$$\prod_{v \in S} |\gamma_v|_v < 2.$$

We see that W is an open neighborhood of 1 in the adelic topology. We claim that $W \cap \mathbb{I}_K^1 \subseteq U$. To see this, note that, for $\gamma \in W \cap \mathbb{I}_K^1$, by definition we have:

$$|\gamma|_{\mathbb{I}_K} = 1 = \prod_{v \in S} |\gamma_v|_v \prod_{v \notin S} |\gamma_v|_v < 2 \prod_{v \notin S} |\gamma_v|_v,$$

where the final inequality follows by our construction of W . Since for every $v \notin S$, we have $\gamma_v \in \mathcal{O}_v$, we have $|\gamma_v|_v \leq 1$, which tells us in particular that $1 < 2|\gamma_v|_v$. This, in turn, implies $|\gamma_v|_v > 1/2$ which then also implies $\gamma_v \in \mathcal{O}_v^\times$ for all $v \notin S$, and so we have our desired containment in U . $\square$

The main result we wish to show is the following.

THEOREM 1.3.10. The norm-1 idele class-group C_K^1 is compact.

To prove this result we will need an adelic version of a classic result in the geometry of numbers. In particular, recall the Blichfeldt-Minkowski lemma that any compact, convex, origin-symmetric set of $\mathbb{R}^n$ with volume greater

than 2^n must contain at least one nonzero integer lattice point. Using the (alternative, depending on your experience) characterization of a lattice as a discrete, co-compact subgroup of a finite-dimensional $\mathbb{R}$ -vector space, in light of Theorem 1.3.1 we are led to the following analogous statement about the diagonal embedding $K \hookrightarrow \mathbb{A}_K$.

LEMMA 1.3.11 (Adelic Blichfeldt-Minkowski). For any $\gamma = (\gamma_v) \in \mathbb{I}_K$, consider the (compact, convex and origin-symmetric!) subset

$$X_\gamma := \{x \in \mathbb{A}_K : |x_v|_v \leq |\gamma_v|_v \ \forall v \in \mathcal{P}_K\}.$$

Then there exists some constant $C > 0$ such that if $|\gamma|_{\mathbb{I}_K} > C$ then $X_\gamma \cap K$ meets $K^\times$.

Proof. Consider the following compact subset:

$$D := \{(d_v) \in \mathbb{A}_K : |d_v|_v \leq 1/4, \ v \mid \infty, \ |d_v|_v \leq 1, \ v < \infty\}.$$

This closed subset is the product of the valuation rings at all the finite places and the balls of radius 1/4 (resp. 1/2) at all the real (resp. complex) places. Thus, under our normalized measure dx on $\mathbb{A}_K$, this has finite and positive measure, since there are only a finite number of infinite places. Moreover, for any $d, d' \in D$ our construction tells us $|d'_v - d_v|_v \leq 1$ for all places v .

Let D_0 be a fundamental domain for K in $\mathbb{A}_K$ (e.g we may choose our construction in the proof of Theorem 1.3.1), which must also have finite positive measure by compactness of the quotient. We now define our constant $C := dx(D_0)/dx(D)$.

Since the idelic absolute value is the module of the multiplication automorphism, we get:

$$|\gamma|_{\mathbb{I}_K} > C \implies dx(\gamma D) > dx(D_0).$$

Thus, the set γD cannot inject into $\mathbb{A}_K/K$ under the quotient map, since it is covered by overlaps with K -cosets of D_0 . It follows γD must contain two distinct elements $\gamma d_1, \gamma d_2$ that project to the same nonzero value in $\mathbb{A}_K/K$, i.e such that $d := \gamma d_1 - \gamma d_2 \in K^\times$ (visibly $\gamma d_1 - \gamma d_2 \neq 0$). Thus, we have:

$$|d_v|_v \leq |\gamma_v|_v |d_{1,v} - d_{2,v}|_v \leq |\gamma_v|_v,$$

and so $d \in X_\gamma$ by definition. Thus, $X_\gamma \cap K^\times$ is non-empty, completing our proof. $\square$

We are now ready to show compactness of C_K^1 .

Proof of 1.3.10. By Lemma 1.3.9, it suffices to find some $W \subseteq \mathbb{A}_K$, compact in the adelic topology, such that the continuous map

$$W \cap \mathbb{I}_K^1 \rightarrow C_K^1$$

is a surjection. To this end, consider a positive real $C > 0$, and pick any $\gamma \in \mathbb{I}_K$ with $|\gamma|_{\mathbb{I}_K} > C$, and any $\beta \in \mathbb{I}_K^1$ and define

$$W_C := \left\{x \in \mathbb{A}_K : |x_v|_v \leq \left|\beta_v^{-1} \gamma_v\right|_v \ \forall v \in \mathcal{P}_K\right\},$$

which is visibly compact in $\mathbb{A}_K$. By the adelic Blichfeldt-Minkowski lemma (1.3.11), we may pick C large enough so that W_C contains a non-zero principal adele $\alpha \in K^\times \subseteq \mathbb{I}_K^1$.

We now set

$$W := \{x \in \mathbb{A}_K : |x_v|_v \leq |\gamma_v|_v \ \forall v \in \mathcal{P}_K\},$$

which is again visibly compact in $\mathbb{A}_K$, and note by construction we have $\alpha\beta \in W$. We deduce $\alpha\beta \in W \cap \mathbb{I}_K^1$, with image $\bar{\beta}$ in C_K^1 . Since $\beta \in \mathbb{I}_K^1$ was arbitrary, we deduce the quotient map is indeed surjective, and thus C_K^1 is compact. $\square$

REMARK. The above result is important for two primary reasons. Firstly, it is akin to the finiteness of the class group of a number field, in the sense that it implies "finiteness" of generalized ideal class groups of number fields. Secondly, as we will see in Chapter II, compactness limits the possibilities of Hecke characters that arise when doing Fourier analysis following the method of Tate.

We close with perhaps the most powerful approximation result on the adeles: strong approximation.

THEOREM 1.3.12. For K a global field, let w be a fixed place and let S be a finite subset of $\mathcal{P}_K \setminus \{w\}$. For every v in S , select arbitrary $a_v \in K$ and $\epsilon_v \in \mathbb{R}_{>0}$. Then there exists some $x \in K$ such that

$$\begin{aligned} |x - a_v|_v &\leq \epsilon_v, \forall v \in S \\ |x|_v &\leq 1, \forall v \notin S \cup \{w\}, \end{aligned}$$

with no restriction on $|x|_w$.

Proof. We may express K as a finite separable of $K_0 = \mathbb{Q}$ or $\mathbb{F}_p(t)$ (which in the positive characteristic case relies on some choice $\mathbb{F}_p(t) \hookrightarrow K$). In the proof of Theorem 1.3.1 we constructed a compact subset C_0 containing a fundamental domain for K_0 in $\mathbb{A}_{K_0}$, which we recall here:

$$C_0 = \left\{ x \in \mathbb{A}_{K_0} : |x_\infty|_\infty \leq \frac{1}{2} \text{ and } |x_v|_v \leq 1, \forall v \neq \infty \right\}.$$

Note again here the notion of the infinite place rests on some choice of separating transcendence basis over $\mathbb{F}_p$ in the positive characteristic case. By adelic base change (Proposition 1.1.9), given a K_0 -basis $\{u_1, \dots, u_n\}$ of K , we have the isomorphism

$$\mathbb{A}_K = \bigoplus_{i=1}^n \mathbb{A}_{K_0} u_i,$$

where the diagonally embedded K on the left side is mapped isomorphically onto diagonally embedded K_0 on the right side. We may then define

$$C := \bigoplus_{i=1}^n C_0 u_i \subseteq \bigoplus_{i=1}^n \mathbb{A}_{K_0} u_i = \mathbb{A}_K$$

Since we have $C_0 \twoheadrightarrow \mathbb{A}_{K_0}/K_0$, by construction we have $C \twoheadrightarrow \mathbb{A}_K/K$, and thus C must contain a fundamental domain for K in $\mathbb{A}_K$. Moreover, since C is a compact subset of $\mathbb{A}_K$, part (b) of Proposition 1.1.3 tells us that it is contained in some set of the form

$$W = \{x \in \mathbb{A}_K : |x_v|_v \leq \delta_v\},$$

where $\delta_v \in \mathbb{R}_{>0}$ is equal to 1 for all but finitely many places.

Now pick some $\gamma \in \mathbb{I}_K$ satisfying the following conditions:

- 1) $|\gamma_v|_v \leq \epsilon_v/\delta_v$, for every $v \in S$;
- 2) $|\gamma_v|_v = 1$, for every $v \in \mathcal{P}_K \setminus (S \cup \{w\})$;

- 3) $|\gamma_w|_w > V \prod_{v \in S} |\gamma_v|_v^{-1}$ for the place w , where V is the "minimal volume" constant associated to $\mathbb{A}_K$ by the adelic Blichfeldt-Minkowski lemma (1.3.11).

Picking such a γ is possible since the conditions imposed satisfy being cofinitely integral everywhere. Lemma 1.3.11 then tells us that $X_\gamma \cap K$ contains a non-zero principal adele z .

Now note, for any $b \in K^\times$, we may write $\mathbb{A}_K = K + bW$, since multiplication by b is an automorphism of K . Thus for an arbitrary $x = (x_v)_v \in \mathbb{A}_K$, we have

$$x = a + y,$$

for some $a \in K$ and $y \in zW$, where z is the principal adele in $X_\gamma \cap K$ discussed earlier in this proof. Thus, we get

$$|x_v - a_v|_v = |y_v|_v \leq \delta_v |z_v|_v \leq \delta_v |\gamma_v|_v,$$

which, by construction, gives us our desired inequality for every place $v \neq w$. $\square$

COROLLARY 1.3.13 (Strong Approximation). For any place $w \in \mathcal{P}_K$, the diagonal embedding

$$K \hookrightarrow \prod_{v \in \mathcal{P}_K \setminus \{w\}} K_v := \mathbb{A}_K^w$$

is dense. The corresponding statement also holds for $K^\times$ and $\mathbb{I}_K^w := \mathbb{I}_K^1 \cap \mathbb{A}_K^w$.

Proof. This is immediate from the preceding theorem; barring a restriction on the place w , we may make the norm of the difference between any given adele and a principal adele arbitrary small in every component. The second statement is a consequence of the fact that $K^\times \subseteq \mathbb{I}_K^1$ and that $\mathbb{I}_K^1$ inherits the subspace topology from $\mathbb{A}_K$ (Lemma 1.3.9). $\square$

Comparing the above result to Theorem 1.3.1 gives us a better appreciation for the discreteness of the embedding $K \hookrightarrow \mathbb{A}_K$; if we remove even a single place from the restricted product the image is dense!

EXAMPLE 1.3.14 (Discrete and Dense). Let K be a number field and let S_∞ be the set of Archimedean places. Ignoring them, we get the so-called **finite adeles**, given by

$$\mathbb{A}_K^\infty := \prod_{v \nmid \infty} K_v.$$

A similar construction gives us $\mathbb{I}_K^\infty$, the **finite ideles**. We remark that the intersection of $K^\times$ with the compact open $\prod_{v \nmid \infty} \mathcal{O}_v$ is exactly the unit group $\mathcal{O}_K^\times$, which is infinite except when $K = \mathbb{Q}$ or is imaginary quadratic.

§1.4 The S-Groups

Recall that if S denotes a finite set of places of a global field K , then the adele ring can be described as the direct limit

$$\mathbb{A}_K = \varinjlim_S \mathbb{A}_{K,S}.$$

It is useful to have "S-analogues" of certain compactness results and volume calculations, as we'll see in the chapter of Tate's thesis. Often (but not always!) we'll be interested in the cases when $S \supseteq S_\infty = \mathcal{P}_{K,\infty}$, i.e when S contains all the infinite places of K .

DEFINITION 1.4.1. Take S any finite set of places of a global field K with $S \supseteq S_\infty$. Then:

(a) For S -nonempty, the S -**adeles** of K are defined as:

$$\mathbb{A}_{K,S} := \prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$$

(b) For S nonempty, the **ring of S -integers** is defined as:

$$\mathcal{O}_{K,S} := K \cap \mathbb{A}_{K,S}$$

i.e the elements of K that are v -integral for all $v \notin S$.

(c) The S -**ideles** are defined as:

$$\mathbb{I}_{K,S} := \prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

By abuse of notation, letting $\mathcal{O}_v^\times$ denote $\{\pm 1\}$ for v real and S^1 for v complex, we may also define the S -ideles (for both number fields and function fields) when $S \not\supseteq S_\infty$, and even when S is empty! This notation will come in handy for computations in §4.3.

(d) The S -**ideles of norm 1** defined as:

$$\mathbb{I}_{K,S}^1 := \mathbb{I}_K^1 \cap \mathbb{I}_{K,S}$$

SANITY CHECK. The S -ideles are algebraically the units of the S -adeles, and $\mathcal{O}_{K,S}^\times = K^\times \cap \mathbb{I}_{K,S}$.

EXAMPLE 1.4.2 (Idelic form of $\text{Cl}(K)$). Let $\mathcal{P}_\infty$ denote the set of infinite places of a number field K . We denote by $\mathbb{A}_{K,\infty}$ (resp. $\mathbb{I}_{K,\infty}$) the open subgroup $\mathbb{A}_{K,\mathcal{P}_\infty} \subseteq \mathbb{A}_K$ (resp. $\mathbb{I}_{K,\mathcal{P}_\infty} \subseteq \mathbb{I}_K$).

By definition, for any idele $(x_v)_v \in \mathbb{I}_K$, we must have that $\text{ord}_v(x_v) = 0$ for cofinitely many places. Thus, if we denote by $\text{Div}(K)$ (see §3.1 for why we use this notation) the set of all fractional ideals of K , we note the map

$$\begin{aligned} \mathfrak{i} : \mathbb{I}_K &\rightarrow \text{Div}(K) \\ (x_v)_v &\mapsto \prod_{v \text{ finite}} \mathfrak{p}_v^{\text{ord}_v(x_v)} \end{aligned}$$

is a well defined surjective group homomorphism with kernel $\mathbb{I}_{K,\infty}$. Moreover, for any principal fractional ideal $\langle x \rangle \leq K$, we note that it arises from the diagonally embedded $x \in \mathbb{I}_K$. Thus, we get the idelic expression of the ideal class group:

$$\text{Cl}(K) \cong \mathbb{I}_K / (K^\times \cdot \mathbb{I}_{K,\infty}). \quad (1.4-1)$$

The primary result of this subsection will be the following theorem, which characterizes the unit group of $\mathcal{O}_{K,S}$ defined above.

THEOREM 1.4.3 (S -unit theorem). For K a global field and any finite set of places $S \supseteq S_\infty$, we have an isomorphism

$$\mathcal{O}_{K,S}^\times \cong \mu(K) \times \mathbb{Z}^{|S|-1},$$

where $\mu(K)$ is the (finite!) group of roots of unity in K .

EXAMPLE 1.4.4 (Dirichlet's Unit Theorem). When K is a number field with r_1 real embeddings and r_2 complex embeddings, then setting $S = S_\infty = \mathcal{P}_{K,\infty}$, Theorem 1.4.3 tells us that $\mathcal{O}_K^\times = \mathcal{O}_{K,S_\infty}^\times$ is finitely generated of rank $r_1 + r_2 - 1$, which is precisely the classical theorem of Dirichlet. Thus we may see 1.4.3 as a generalization of this result, of which we will proceed to give a neat proof using the adelic formulation.

To prove the S -unit theorem we will need a few intermediary results, the first of which is the following lemma.

LEMMA 1.4.5. For K a global field, $x \in K$ is a root of unity (i.e has finite multiplicative order) if and only if $|x|_v = 1$ for all $v \in \mathcal{P}_K$.

Proof. One direction is clear, so for the other one suppose x has norm 1 at all places.

When K is a number field, this is a classical result of Kronecker, but we recount a detailed proof here. We note x must be an algebraic integer, all of whose complex embeddings have norm 1. Its minimal polynomial $f(t)$ is integral and has the form $\prod_i (t - x_i)$ taken over all Galois conjugates of x . Consider the coefficient of t^k . It is the sum of $\binom{n}{k}$ terms, where n is the degree of f and where each term is a product of x_i 's, and thus must have complex norm 1. Thus the coefficient, since it lies in $\mathbb{Z}$, must be bounded in absolute value by $\binom{n}{k}$. Since $x \in \mathcal{O}_K$, we know $n \leq [K : \mathbb{Q}]$. Moreover, for any $\sigma \in \text{Gal}(K/\mathbb{Q})$, we note that $|\sigma(x^m)| = |\sigma(x)|^m = 1$ for all $m \geq 1$. Thus the minimal polynomials of all such x^m have coefficients bounded by $\binom{[K:\mathbb{Q}]}{\lfloor [K:\mathbb{Q}]/2 \rfloor}$ (i.e the largest binomial coefficient), and their degree is also bounded by $[K : \mathbb{Q}]$. Thus the powers of x are roots of a finite set of polynomials, so the cyclic group generated by x is finite, making it a root of unity

When K has positive characteristic with field of constants $\mathbb{F}_q$, we can associate it with the function field of some smooth proper curve X geometrically connected over $\mathbb{F}_q$. The condition of having norm 1 at all places then tells us that x is a rational function on X with no zeroes or poles, hence lies in the field of constants $\mathbb{F}_q$, and thus has finite order. $\square$

Next, we need a generalization of the compactness of the norm-1 idele class group C_K^1 to the S -ideles.

LEMMA 1.4.6. The quotient $\mathbb{I}_{K,S}^1 / \mathcal{O}_{K,S}^\times$ is compact, for any finite (possibly empty!) set S .

Proof. Firstly $\mathcal{O}_{K,S}^\times$ is visibly a subgroup of $\mathbb{I}_{K,S}^1$, using the fact that $K^\times$ embeds in $\mathbb{I}_K$ and the product formula (1.3.6), thus the quotient makes sense. We also see that $\mathbb{I}_{K,S}$ is by definition an open subgroup of $\mathbb{I}_K$, and thus $\mathbb{I}_{K,S}^1$ is open (hence closed!) in $\mathbb{I}_K^1$ under the subspace topology. Since C_K^1 is a compact (Hausdorff) quotient, it follows $\mathbb{I}_{K,S}^1 / \mathcal{O}_{K,S}^\times$ is a closed subgroup, and thus also compact. $\square$

We now are ready to prove the main result.

Proof of 1.4.3. Consider the "adelic circle", defined as the product of all the norm-1 elements of each completion, i.e.

$$C_A := \prod_{v \in \mathcal{P}_K} C_v^1,$$

where $C_v^1 := \{x_v : |x_v|_v = 1\} \subseteq K_v^\times$. Note when v is non-Archimedean then $C_v^1 = \mathcal{O}_v^\times$. By Tychonoff, this is a compact subgroup of $\mathbb{I}_{K,S}$ for any S . Thus we have the (topologically!) exact sequence:

$$1 \rightarrow C_{\mathbb{A}} \rightarrow \mathbb{I}_{K,S} \rightarrow \prod_{v \in S} (K_v^\times / C_v^1) \rightarrow 1. \quad (1.4-2)$$

Note that when v is non-Archimedean then

$$K_v^\times / C_v^1 = K_v^\times / \mathcal{O}_v^\times \cong \mathbb{Z},$$

and when v is Archimedean we have

$$K_v^\times / C_v^1 \cong \mathbb{R}_{>0} \cong \mathbb{R}.$$

Thus, if we have n non-Archimedean places and m Archimedean places in S , (1.4-2) becomes

$$1 \rightarrow C_{\mathbb{A}} \rightarrow \mathbb{I}_{K,S} \rightarrow \mathbb{Z}^n \times \mathbb{R}^m \rightarrow 1. \quad (1.4-3)$$

Now we analyse how $K^\times \hookrightarrow \mathbb{I}_K$ meets the objects in the exact sequence in (1.4-3). Firstly, we see $K^\times \cap C_{\mathbb{A}}$ consists of all elements that have norm 1 at every place, which Lemma 1.4.5 tells us is precisely $\mu(K)$. For the middle object, we recover exactly $\mathcal{O}_{K,S}^\times$. For the final object, let L denote the image of $K^\times \cap \mathbb{I}_{K,S} =: \mathcal{O}_{K,S}^\times$ under the quotient map. This transforms (1.4-3) to the following short exact sequence:

$$1 \rightarrow \mu(K) \rightarrow \mathcal{O}_{K,S}^\times \rightarrow L \rightarrow 1. \quad (1.4-4)$$

It remains to show that $L \cong \mathbb{Z}^{n+m-1}$. Consider the natural continuous map

$$\begin{aligned} \phi : \prod_{v \in S} (K_v^\times / C_v^1) &\rightarrow |\mathbb{I}_K|_{\mathbb{I}_K} \\ (x_v)_{v \in S} &\mapsto \prod_{v \in S} |x_v|_v. \end{aligned}$$

When K is a number field, this can abstractly be seen as a continuous homomorphism $\mathbb{Z}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_{>0}$ that is nontrivial on each copy of $\mathbb{R}$, whose kernel is thus isomorphic to $\mathbb{Z}^n \times \mathbb{R}^{m-1}$. Since the image L of $\mathcal{O}_{K,S}^\times$ lies within this kernel, it is in fact a discrete and cocompact subgroup. Any (nontrivial) subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$, and any discrete subgroup of $\mathbb{R}^{m-1}$ must be finite free abelian, and the cocompactness conditions enforces a rank $m-1$ condition as well. Thus, $L \cong \mathbb{Z}^{n+m-1}$, as desired.

When K has positive characteristic, recalling the image of the idelic norm is some finite index subgroup of $q^{\mathbb{Z}}$ (hence isomorphic to $\mathbb{Z}$), we can view ϕ as a surjective map $\mathbb{Z}^n \rightarrow \mathbb{Z}$ (recall a function field has no Archimedean places!) whose kernel thus must be isomorphic to $\mathbb{Z}^{n-1}$, as desired. $\square$

The above result motivates the following definition.

DEFINITION 1.4.7. For S any nonempty finite set of places of a global field K containing the infinite ones, the **S -class group** of K is

$$C_{K,S} := \mathbb{I}_K / (K^\times \cdot \mathbb{I}_{K,S})$$

OBSERVATION. We claim the following inclusion:

$$\mathbb{I}_K^1 / (K^\times \cdot \mathbb{I}_{K,S}^1) \hookrightarrow \mathbb{I}_K / (K^\times \cdot \mathbb{I}_{K,S}) =: C_{K,S}.$$

has finite index, and is in fact an isomorphism in the number field case! When K has characteristic zero, any

element of the right side we may always represent it by an idele of norm 1 by adjusting the value in K_v for any infinite place v . When K is a function field we may adjust the value in K_v for some $v \in S$ to generate some finite index subgroup of the value group of the idelic norm, and thus it follows we get a finite index image of the above inclusion. Putting these observations, along with previous results, together, we get the following result.

PROPOSITION 1.4.8 (Finiteness of S -Class Groups). Let S be any nonempty finite set of places of a global field K containing the infinite places. Then $C_{K,S}$ is a finite group.

Proof. Follows immediately from the above observation and the proof of Theorem 1.4.3. □

REMARK. Should we allow S to be empty in the above proposition statement (only possible in positive characteristic), we would get that $C_{K,\emptyset}$ is the direct product of $\mathbb{Z}$ with a finite group, a (useless, but perhaps mildly amusing) fact we leave the reader to check.

The preceding result will be crucial in volume and residue calculations of L -functions over number fields, which will be covered in the subsequent chapter.

EXAMPLE 1.4.9 (Finiteness of the ideal class group). When $S = S_\infty$ for K a number field, we see that the quotient in (1.4-1) from Example 1.4.2 is precisely

$$C_{K,S_\infty} = \text{Cl}(K).$$

Proposition 1.4.8 then is an idelic formulation of the fact that the ideal class group of a number field is finite. When one unravels the details, the adelic proof of finiteness of the ideal class group is essentially the same as the classical one, only packaged differently. The advantage of the adelic packaging lies in its versatility and in the uniformity of arguments that one can apply to tackle all global fields at once, as we will see throughout this text.

Chapter II

THE THESIS OF TATE (A.K.A, GL_1)

"In [his thesis], Tate provides an elegant and unified treatment of the analytic continuation of the L -functions attached by Hecke to his Größencharaktere... The power of the methods of abelian harmonic analysis in the setting of Chevalley's adeles/ideles provided a remarkable advance over the classical techniques used by Hecke."

Ch. 6 of *An Introduction to the Langlands Program*

STEPHEN S. KUDLA

One can generalize classical Dirichlet characters to quasi-characters of ray class groups of number fields, and then one can generalize even further to continuous characters of the idele class group C_K for K a global field. It is a famous result of Hecke that $L(s, \chi)$ has analytic continuation and satisfies an appropriate functional equation for any number field K and any idele class character χ . The original formulation of this result involved a complicated computation with theta functions and Mellin transforms. Moreover in the functional equation a so-called *root number* $W(\chi)$ associated to each quasi-character χ kept popping up, with no good explanation for its value.

In his 1950 PhD thesis John Tate made use of Fourier analysis on adèle groups to reprove the results of Hecke, first by establishing local functional equations at all the completions of a global field and then using the adelic formulation to give an elegant description of a global functional equation on quasi-characters of the idele group. We do not discuss Hecke's original approach here, but we recommend the interested reader to peruse Chapter VII of [NS10] for a detailed exposition on that matter in order to gain a better appreciation of the relative elegance of Tate's packaging.

Section 2 covers the necessary local theory, the results of which will be patched together in subsequent sections. Section 3 sets up the theory of adelic Fourier analysis (with a quick detour on some Riemann-Roch) and proves the global functional equation for the zeta integral. Section 4 then discusses some concrete examples and applications of Tate's approach in the theory of L -functions and Dedekind zeta functions.

The unacquainted reader may also question why the moniker of " GL_1 " for this chapter. The answer lies in the deeper theory of adelic automorphic forms, and their associated representation theory, of which the idelic analysis may be viewed as a special case. At some point in the (much later) future the author will likely exposit upon this in an expansion of this text.

SECTION 2

THE LOCAL THEORY

§2.1 The Local Functional Equations

In his thesis, Tate worked with some ad-hoc function spaces over local and global fields. We standardize our workings with the so-called *Schwartz-Bruhat functions* on local fields. Over an Archimedean field (i.e $\mathbb{R}$ or $\mathbb{C}$), recall a **Schwartz** function (more commonly known as a *rapidly-decreasing function*) is a smooth $\mathbb{C}$ -valued function f such that

$$\lim_{|x| \rightarrow \infty} p(x) D^k f(x) = 0$$

for all polynomials $p(x)$ and all k -th derivative operators D^k (corresponding to d^k / dx^k for $F = \mathbb{R}$ and $\partial^k / (\partial x^i \partial y^{k-i})$, for all $i \in [0, k]$ for $F = \mathbb{C}$). These functions classically form a useful space on which to do Fourier analysis, and so we define a similarly useful generalization for our adelic purposes.

DEFINITION 2.1.1. A **Schwartz-Bruhat** function on a local field F is a Schwartz function if F is Archimedean and a smooth (i.e locally constant, when F is non-Archimedean), compactly-supported function if F is non-Archimedean. We denote by $\mathcal{S}(F)$ the complex vector space of Schwartz-Bruhat functions on F .

Before analyzing the global zeta integral of a function on an adèle group of a global field, we first understand the induced zeta functions on each of the local components. The goal of this subsection will be to describe a functional equation of zeta functions on a local field by introducing so-called *local L -factors* associated to a character χ (and its dual $\chi^\vee$ under the Fourier transform) of a local field. In particular, these local L -factors will be realized as the gcd of some local zeta functions. We start with a slew of notations and definitions.

When F is non-Archimedean, let $\mathcal{O}_F$ denote its ring of integers, $\mathfrak{p}$ its maximal ideal with uniformizer π and residue field $\mathbb{F}_q$. For any local field, we may decompose its multiplicative group of units as

$$F^\times \cong \mathcal{O}_F^\times \times \mathcal{S}_F$$

where $\mathcal{O}_F^\times$ is the group of norm-1 elements of F and $\mathcal{S}_F := |F^\times|$, i.e. the image of $F^\times$ in the valuation map, up to choice of uniformizer π and homomorphic retraction $F^\times \rightarrow \mathcal{O}_F^\times$. In the non-Archimedean case the group $\mathcal{O}_F^\times$ is the literal group of units of the valuation ring, and in the Archimedean case we are abusing notation a little here. If $F = \mathbb{R}$ then $\mathcal{O}_F^\times := \{\pm 1\}$ and if $F = \mathbb{C}$ then $\mathcal{O}_F^\times := S^1$. This decomposition allows us to give a nice characterization of the multiplicative quasi-characters of a local field.

PROPOSITION 2.1.2. For F a local field, any quasi-character of $F^\times$ is of the form

$$\chi : x \mapsto \mu(\tilde{x}) |x|_F^s,$$

where μ is a character (i.e S^1 -valued homomorphism) of $\mathcal{O}_F^\times$, $\tilde{x}$ denotes the image of x under the (choice of) homomorphic retraction $F^\times \rightarrow \mathcal{O}_F^\times$, and $s \in \mathbb{C}$.

Proof. Let μ be the restriction of χ to $\mathcal{O}_F^\times$. Since the subgroup $\mathcal{O}_F^\times$ is compact and $\mathbb{C}^\times$ has no "small subgroups", it

follows the image of μ must lie in S^1 , and hence it must be a character. We note that the map

$$x \mapsto \chi(x)/\mu(\tilde{x})$$

is by construction trivial on $\mathcal{O}_F^\times$, hence it must factor through the map

$$|\cdot|_F : F^\times \rightarrow \mathcal{S}_F.$$

But the value group $\mathcal{S}_F$ is either isomorphic to $\mathbb{R}_{>0}$ (if F is Archimedean) or to $\mathbb{Z}$ (if F is non-Archimedean), and the only quasi-characters on these groups are of the form $t \mapsto t^s$. Thus χ must decompose as described above. $\square$

DEFINITION 2.1.3. For any quasi-character $\chi = \mu |\cdot|^s$ of $(F^\times)$, we call $\Re(s)$ the **exponent** of χ . Furthermore, we say χ is **unramified** if $\chi(\mathcal{O}_F^\times) = 1$ (i.e if $\mu = 1$). We denote by $\mathcal{X}(F^\times)$ the total space of quasi-characters on $F^\times$.

WARNING. Recall that for any $q \in \mathbb{R}_{>0}$ we have the formula:

$$q^i = \cos(\log(q)) + i \sin(\log(q)).$$

In particular, in the non-Archimedean case the value of s is *not* uniquely determined (e.g $s + (2\pi i N)/\log(q)$ give the same χ for all $N \in \mathbb{Z}$), which is why we define the exponent to be the real part, which is uniquely determined. Thus we may abuse notation to write $\Re(\chi)$ without confusion.

We may consider the equivalence relation on $\mathcal{X}(F^\times)$ given by declaring two quasi-characters to be equivalent if their quotient is unramified. By our observations above, we see that any such equivalent class consists of quasi-characters of the form,

$$\chi(x) = \chi_0(x) |x|_F^s,$$

where χ_0 is a representative of the equivalence class, and $s \in \mathbb{C}$ is variable. This then allows us to give each equivalence class the structure of a Riemann surface, since the variable s determines the character. By our observation above, each $s \in \mathbb{C}$ determines a distinct character when F is Archimedean, and in the non-Archimedean case the character is only determined modulo $2\pi i/\log(q)$. In particular, each equivalence class is isomorphic to $\mathbb{C}$ if F is Archimedean and to the infinite cylinder $\mathbb{C}/(2\pi i\mathbb{Z}/\log(q))$ otherwise. Thus our total space $\mathcal{X}(F^\times)$ is disjoint union of (at most countably many) Riemann surfaces, so for a function with such quasi-characters as an input we may reasonably discuss notions such as holomorphicity, convergence and analytic continuation, with the caveat that such details will have to be worked out for each connected component.

REMARK (Terminology). The reasoning behind the term "unramified" character is as follows: if F is a non-Archimedean local field with maximal abelian extension F^{ab} , then any character χ of $\text{Gal}(F^{\text{ab}}/F)$ factors through the Galois group of some finite abelian extension L/F . Naturally, the inertia subgroup $I_{F^{\text{ab}}/F}$ of $\text{Gal}(F^{\text{ab}}/F)$ maps onto the inertia subgroup of $\text{Gal}(L/F)$, which is trivial if and only if L/F is an unramified field extension. Thus we may reasonably call χ unramified if $\chi|_{I_{F^{\text{ab}}/F}} = 1$.

If $\theta : F^\times \hookrightarrow \text{Gal}(F^{\text{ab}}/F)$ denotes the local Artin homomorphism, then it is known that $\mathcal{O}_F^\times$ maps isomorphically onto $I_{F^{\text{ab}}/F}$. If we denote

$$\eta := \chi \circ \theta : F^\times \rightarrow \mathbb{C}^\times,$$

it follows $\eta|_{\mathcal{O}_F^\times} = 1$ if and only if L/F is unramified. This is exposited in more detail in Chapter 7 of [Chi09].

We now state certain *local* L -functions that will show up in the functional equation involving the local zeta integral, which will be described later.

(1) In the non-Archimedean case, we set:

$$L(\chi) := \begin{cases} (1 - \chi(\pi_F))^{-1}, & \text{if } \chi \text{ is unramified,} \\ 1, & \text{otherwise.} \end{cases} \quad (2.1-1)$$

(2) In the Archimedean case we recall by Gelfand-Mazur that $F = \mathbb{C}$ or $\mathbb{R}$.

(i) If $F = \mathbb{C}$, note any χ is of the form:

$$\chi_{s,n} : z \mapsto |z|^s \left(\frac{z}{|z|} \right)^n$$

for $s \in \mathbb{C}$ and $n \in \mathbb{Z}$ uniquely defined. Here we use the classical fact that the Pontryagin dual of S^1 is $\mathbb{Z}$. Also note by $|z|$ here we mean the usual absolute value on $\mathbb{C}$ (i.e. not our normalized one!). We then set:

$$L(\chi_{s,n}) := \Gamma_{\mathbb{C}}\left(s + \frac{|n|}{2}\right) := (2\pi)^{-\left(s + \frac{|n|}{2}\right)} \Gamma\left(s + \frac{|n|}{2}\right). \quad (2.1-2)$$

(ii) If $F = \mathbb{R}$ then $U_F := \{\pm 1\}$. Here any quasi-character is of the form $\chi = \mu |\cdot|^s$, where μ and s are uniquely defined, with μ either trivial or the sign homomorphism $\text{sgn} : x \mapsto x/|x|$. We then set:

$$L(\chi) := \begin{cases} \Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2), & \mu = 1, \\ \Gamma_{\mathbb{R}}(s+1), & \mu = \text{sgn}. \end{cases} \quad (2.1-3)$$

DEFINITION 2.1.4. The functions $\chi \mapsto L(\chi)$ described in (2.1-3), (2.1-2) and (2.1-1) are the **local L -factors** of their corresponding local fields. We also introduce the following convenient notation:

$$L(s, \chi) := L(\chi |\cdot|^s).$$

Moreover we define the **shifted dual** (or sometimes **twisted dual**) of χ by:

$$\chi^\vee := \chi^{-1} |\cdot|.$$

OBSERVATION. The twisted dual satisfies the following nice identities, which will come of use later:

$$\Re(\chi^\vee) = 1 - \Re(\chi),$$

and

$$L(s, \chi^\vee) = L(1-s, \chi^{-1}).$$

In order to discuss the local zeta integral, we must establish a sound measure-theoretic basis with which we will integrate. In Tate's thesis (which only treats number fields) he normalizes his Haar measures on the local field to be self-dual relative to a choice of nontrivial additive character ψ , from which he obtains the nice classical Fourier identity $f(x) = \widehat{\widehat{f}}(-x)$. Here instead we explore the functional equation's dependency on choice of additive character and Haar measure.

Given dx a Haar measure on F , define

$$d^\times x := c \cdot \frac{dx}{|x|} \quad (2.1-4)$$

for some $c \in \mathbb{R}_{>0}$, which we always set to 1 in the Archimedean case. Then $d^\times x$ is a Haar measure on $F^\times$. One may ask why we don't always set $c = 1$, and the reason for allowing ourselves this flexibility will become clear when we get to the computations of the global zeta functions in §3.2.

Fixing a nontrivial additive character $\psi \in \widehat{F}$, for any $f \in \mathcal{S}(F)$ we define its **Fourier transform** by:

$$\widehat{f}(y) := \int_F f(x) \psi(xy) dx. \quad (2.1-5)$$

The function $\widehat{f}$ is well-defined and in fact again lies in $\mathcal{S}(F)$, though it depends on our choice of ψ and dx . When $F = \mathbb{R}$ or $\mathbb{C}$ this fact is classical; when F is non-Archimedean this follows by a simple generalization of the proof of Proposition 1.2.14. In particular, one can the fact that any such f is compactly supported and locally constant (hence some finite linear combination of the function used for computation in that proof), and that changing the measure or additive character ψ only affects things by some global scaling factor. We leave the (somewhat tedious) verification of the details to the reader.

We are now ready to define the (local!) zeta function.

DEFINITION 2.1.5. Given $f \in \mathcal{S}(F)$ and $\chi \in \mathcal{X}(F^\times)$, we define the associated **local zeta integral** to be:

$$Z(f, \chi) := \int_{F^\times} f(x) \chi(x) d^\times x.$$

The convenience of defining the space $\mathcal{X}(F^\times)$ in Definition 2.1.3 is hopefully clear with the L -factors and zeta functions defined, as it allows us to write zeta functions, a priori taking quasi-characters χ as inputs, as functions on $\mathbb{C}$, which is frequently used in literature for ease of notation. The following is the main result of this subsection.

THEOREM 2.1.6 (The Local Functional Equation). For F a local field, take $f \in \mathcal{S}(F)$ and $\chi = \mu |\cdot|^s \in \mathcal{X}(F^\times)$ with exponent $\sigma = \Re(s)$. Then:

- (a) If $\sigma > 0$ then the local zeta integral $Z(f, \chi)$ is absolutely convergent and holomorphic in such χ .
- (b) If $\sigma \in (0, 1)$ then the following functional equation holds:

$$Z(\widehat{f}, \chi^\vee) = \gamma(\chi, \psi, dx) Z(f, \chi)$$

for some $\gamma(\chi, \psi, dx) \in \mathbb{C}^\times$ independent of f , a priori defined on the strip $\sigma \in (0, 1)$ on each connected component of $\mathcal{X}(F^\times)$, and for all $\chi \in \mathcal{X}(F^\times)$ via meromorphic continuation.

- (c) There exists $\epsilon(\chi, \psi, dx) \in \mathbb{C}^\times$ satisfying:

$$\gamma(\chi, \psi, dx) = \epsilon(\chi, \psi, dx) \frac{L(\chi^\vee)}{L(\chi)}$$

as a meromorphic function of $\chi \in \mathcal{X}(F^\times)$.

OBSERVATION. Note that part (a) tells us that $Z(f, \chi)$ is absolutely convergent to the right of $\sigma = 0$ on each connected component of $\mathcal{X}(F^\times)$, from which it follows by construction that $Z(\widehat{f}, \chi^\vee)$ is absolutely convergent

to the left of $\sigma = 1$. Using this domain overlap along with part (b), we see that this theorem automatically yields a meromorphic continuation of $Z(f, \chi)$ to all $\chi \in \mathcal{X}(F^\times)$.

Proof of part (a) in Theorem 2.1.6. To show $Z(f, \chi)$ is absolutely convergent when $\sigma > 0$, using our relationship between our Haar measures in (2.1-4) and the fact that μ is unitary it suffices to show that:

$$I(f, \sigma) = c \int_{F \setminus \{0\}} |f(x)| |x|^{\sigma-1} dx < \infty,$$

which will also ensure holomorphicity in χ . In the Archimedean case, finiteness of $I(f, \sigma)$ follows from the fact that f is a Schwartz function, and thus rapidly decaying as $|x|$ approaches infinity, and that $|x|^{\sigma-1}$ is integrable near 0^1 , so we are happy.

In the non-Archimedean case, f is locally constant with compact support, so it must factor through some finite group quotient of the form $\mathfrak{p}^n / \mathfrak{p}^m$, for some integers $m \leq n$ (here $\mathfrak{p}^k = \langle \pi^k \rangle$ represents the fractional ideal of $\mathcal{O}_F$ generated by the k -th power of its uniformizer). By linearity and translation-invariance of the Haar measure, it suffices to check finiteness of the integral for functions f that are just the characteristic functions of the various ideals $\mathfrak{p}^j$.

Using unique factorization in any fractional ideal, we can decompose it into a disjoint union:

$$\mathfrak{p}^j \setminus \{0\} = \bigsqcup_{k=j}^{\infty} \pi^k \mathcal{O}_F^\times.$$

Thus our integral becomes:

$$\begin{aligned} I(f, \sigma) &= c \int_{F \setminus \{0\}} |f(x)| |x|^{\sigma-1} dx \\ &= \int_{F^\times} |f(x)| |x|^\sigma d^\times x \\ &= \sum_{k \geq j} \int_{\pi^k \mathcal{O}_F^\times} |x|^\sigma d^\times x \\ &= \sum_{k \geq j} \text{Vol}(\mathcal{O}_F^\times, d^\times x) q^{-k\sigma} \\ &= \text{Vol}(\mathcal{O}_F^\times, d^\times x) \frac{q^{-j\sigma}}{1 - q^{-\sigma}} \end{aligned}$$

which is finite when $\sigma > 0$, as desired. □

To prove the next part of our main theorem, we use a lemma that is key in Tate's thesis.

LEMMA 2.1.7. For $g \in \mathcal{S}(F)$ and any quasi-character χ with exponent $\sigma \in (0, 1)$, we have:

$$Z(f, \chi) Z(\widehat{g}, \chi^\vee) = Z(\widehat{f}, \chi^\vee) Z(g, \chi).$$

Proof. Expanded out, the left side of the Lemma is the following product of integrals:

$$Z(f, \chi) Z(\widehat{g}, \chi^\vee) = \int_{F^\times} f(x) \chi(x) d^\times x \cdot \int_{F^\times} \widehat{g}(y) \chi^{-1}(y) |y| d^\times y,$$

¹For $F = \mathbb{C}$, we should be a bit careful here since the Haar measure and absolute value here are not the usual ones, but we note that we have $\frac{|dz \wedge d\bar{z}|}{|z|^{2(\sigma-1)}} = \frac{|dz|}{|z|^{\sigma-1}} \times \frac{d\bar{z}}{|z|^{\sigma-1}}$, so this integral remains finite.

both of which are absolutely convergent by the settle part (a) of Theorem 2.1.6. We can turn this into a double integral over the product space by:

$$\iint_{F^\times \times F^\times} f(x) \widehat{g}(y) \chi(xy^{-1}) |y| \, d^\times x \, d^\times y.$$

Using invariance of the product measure under the "shearing" automorphism $(x, y) \mapsto (x, xy)$, we obtain:

$$\iint_{F^\times \times F^\times} f(x) \widehat{g}(xy) \chi(y^{-1}) |xy| \, d^\times x \, d^\times y,$$

which, by Fubini's theorem, is equal to:

$$\int_{F^\times} \left(\int_{F^\times} f(x) \widehat{g}(xy) |x| \, d^\times x \right) \chi(y^{-1}) |y| \, d^\times y.$$

To prove the lemma it remains to show that the inner integral is symmetric in f and g . Expanding out the Fourier transform of g (recalling our convention in (2.1-5) and transferring it to an integral over F using (2.1-4), we get that the inner integral is equal to:

$$c \iint_{F \times F} f(x) g(z) \psi(xyz) \, dz \, dx.$$

Note we replaced $F^\times$ with F , which is of no issue since $\{0\} \subseteq F$ is a set of measure zero. We switch the order of integration via Fubini to obtain:

$$c \iint_{F \times F} f(x) g(z) \psi(xyz) \, dx \, dz = c \int_F g(z) \widehat{f}(zy) \, dz = \int_{F^\times} g(z) \widehat{f}(zy) |z| \, d^\times z$$

which gives us the desired result. $\square$

Proof of (b) in 2.1.6. Let χ be a representative of any connected component of $\mathcal{X}(F^\times)$. First suppose $Z(f, \chi) \neq 0$. Then our functional equation in Lemma 2.1.7 tells us that:

$$\gamma(\chi, \psi, dx) := \frac{Z(\widehat{f}, \chi^\vee)}{Z(f, \chi)}$$

is independent of our choice of $f \in \mathcal{S}(F)$, when $Z(f, \chi) \neq 0$. Note that:

$$\chi^\vee = (\mu |\cdot|^s)^\vee = \mu^{-1} |\cdot|^{1-s}$$

By (the already proved!) part (a) of Theorem 2.1.6, we know $Z(\widehat{f}, \chi^\vee)$ is absolutely convergent when $\sigma = \Re(s) < 1$, and we have already seen that the denominator of γ is absolutely convergent for all positive σ , so the functional equation in terms of γ holds for all $\sigma \in (0, 1)$ when $Z(f, \chi) \neq 0$.

If $Z(f, \chi) = 0$, we want to show that $Z(\widehat{f}, \chi^\vee) = 0$ as well, which will follow from Lemma 2.1.7 so long as we exhibit, for any χ , the existence of a $g \in \mathcal{S}(F)$ so that $Z(g, \chi) \neq 0$. For this, we turn to the proof of part (c) of the theorem, where we will compute explicit expressions for the zeta function and for the γ -factor which will directly imply these results. These are tabulated in Tables 2.2.A and 2.2.B in §2.2.

Finally, the fact that γ admits meromorphic continuation to all of $\mathcal{X}(F^\times)$ will follow from the fact that it will be a familiar complex-valued function on the strip $\Re(\chi) \in (0, 1)$ on each connected component, and thus will trivially admit a meromorphic continuation to all of $\mathcal{X}(F^\times)$. Again, these are tabulated and discussed further in §2.2. $\square$

Proof of part (c) in Theorem 2.1.6 in the Archimedean case. As shown in the proof of part (b) of the theorem, the value of the γ function is independent of our choice of $f \in \mathcal{S}(F)$. Thus we may choose a special function,

or family of functions, for each of the possible cases of F to make our computations convenient. We will select a standard measure $\mathrm{d}x$ for each case, which is self-dual for an appropriate choice of ψ , and then analyze dependency on arbitrary $\mathrm{d}x$ and ψ afterward, in §2.2.

Case 1: $F = \mathbb{R}$

We take $\mathrm{d}x$ to be the usual Lebesgue measure, and set our additive character to be:

$$\psi(x) = e^{-2\pi i x}.$$

As we noted in our discussion of the local L -factors, any $\chi \in \mathcal{X}(\mathbb{R}^\times)$ is of the form $|\cdot|^s$ or $\text{sgn}|\cdot|^s$.

Sub-Case (i): $\chi = |\cdot|^s$

Take $f(x) = e^{-\pi x^2}$, which is visibly in $\mathcal{S}(\mathbb{R})$. Then recalling that we set the scaling constant in the measure transformation of (2.1-4) to 1, we get:

$$Z(f, \chi) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \mathrm{d}^\times x = 2 \int_0^\infty e^{-\pi x^2} x^{s-1} \mathrm{d}x.$$

Using a substitution $u = \pi x^2$ (hence $\mathrm{d}u = 2\pi x \mathrm{d}x$), the integral becomes:

$$\pi^{-s/2} \int_0^\infty e^{-u} u^{s/2-1} \mathrm{d}u = \pi^{-s/2} \Gamma(s/2).$$

and we see from (2.1-3) that this is exactly our local L -factor, so in fact $Z(f, \chi) = L(\chi)$ for all characters of this form. Classical Fourier analysis also tells us f is self-dual via the Fourier transform, i.e.

$$\widehat{f}(y) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i xy} \mathrm{d}x = f(x).$$

Thus, we have:

$$Z(\widehat{f}, \chi^\vee) = \int_{\mathbb{R}^\times} \widehat{f}(x) \chi^\vee(x) \mathrm{d}^\times x = \int_{\mathbb{R}^\times} f(x) \chi^\vee(x) \mathrm{d}^\times x = Z(f, \chi^\vee)$$

which is equal to $L(\chi^\vee)$ by our previous calculation in this proof. Thus if $\chi = |\cdot|^s$, we have:

$$\begin{aligned} \gamma(\chi, \psi, \mathrm{d}x) &= \frac{L(\chi^\vee)}{L(\chi)} \\ \implies \epsilon(\chi, \psi, \mathrm{d}x) &= 1. \end{aligned} \tag{2.1-6}$$

Note here that division by $L(\chi) = \Gamma_{\mathbb{R}}(s)$ is no issue, since the $\Gamma_{\mathbb{R}}$ is (globally) non-vanishing and well-defined on the strip $\Re(s) \in (0, 1)$.

Sub-Case (ii): $\chi = \text{sgn}|\cdot|^s$

In this case we choose $f(x) = xe^{-\pi x^2}$, which is again immediately verifiable as rapidly decaying. Expanding the local zeta function we get:

$$\begin{aligned} Z(f, \chi) &= \int_{\mathbb{R}^\times} x e^{-\pi x^2} \frac{x}{|x|} |x|^s \mathrm{d}^\times x \\ &= \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^{s+1} \mathrm{d}^\times x \\ &= \pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right). \\ &= \Gamma_{\mathbb{R}}(s+1) \end{aligned}$$

which we again recall from (2.1-3) as by definition our real local L -factor $L(\chi)$.

Classical Fourier duality on the Schwartz space tells us that multiplication by i corresponds to multiplication by x , so we get the identity

$$\widehat{f}(y) = iy e^{-\pi y^2} = i f(y),$$

from which it follows

$$Z(\widehat{f}, \chi^\vee) = \int_{\mathbb{R}^\times} i f(y) \chi^\vee d^\times x = i L(\chi^\vee).$$

Thus, by a similar calculation to the last sub-case we deduce:

$$\epsilon(\chi, \psi, dx) = i. \quad (2.1-7)$$

Case 2: $F = \mathbb{C}$

Here we let our measure on $\mathbb{C}$ be $|dz \wedge d\bar{z}| = 2 |dx \wedge dy|$, double the standard Lebesgue measure. In polar coordinates this is $2r |dr \wedge d\theta|$, which also implies the Haar measure on $F^\times$ is $d^\times z = (2/r) dr d\theta$ (we divide by r^2 since our normalized absolute value on $\mathbb{C}$ is the square of the usual one). We choose this as it is self-dual with respect to the additive quasi-character

$$\psi(z) := e^{-2\pi i(z+\bar{z})}.$$

We also adjust the standard norm on $\mathbb{C}$ to agree with the modulus, i.e

$$|z| := z\bar{z}.$$

Since $\mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1$, any multiplicative quasi-character must be of the form:

$$\chi_{s,n} := z \mapsto |z|^s \left(\frac{z}{|z|} \right)^n$$

for $s \in \mathbb{C}$ and $n \in \mathbb{Z}$ uniquely defined. We choose our class of functions to be:

$$f_n(z) := \begin{cases} \frac{1}{2\pi} (\bar{z})^n e^{-2\pi z\bar{z}}, & n \geq 0, \\ \frac{1}{2\pi} z^{-n} e^{-2\pi z\bar{z}}, & n < 0. \end{cases}$$

We note that f is visibly a Schwartz function for any n . Computing the local zeta function for non-negative n , we get:

$$\begin{aligned} Z(f_n, \chi_{s,n}) &= \int_{\mathbb{C}^\times} f_n(z) \chi_{s,n}(z) d^\times z \\ &= \frac{1}{2\pi} \int_{\mathbb{C}^\times} (\bar{z})^n e^{-2\pi z\bar{z}} (z\bar{z})^s e^{in \arg(z)} d^\times z \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty (re^{-i\theta})^n e^{-2\pi r^2} (r^2)^s e^{in\theta} (2/r) dr d\theta \\ &= \frac{1}{\pi} \int_0^{2\pi} \int_0^\infty r^{2s+n-1} e^{-2\pi r^2} dr d\theta \\ &= (2\pi)^{-(s+\frac{n}{2})} \int_0^\infty e^{-2\pi r^2} (2\pi r)^{s+\frac{n}{2}-1} 4\pi r dr \\ &= (2\pi)^{-(s+\frac{n}{2})} \int_0^\infty e^{-t} t^{s+\frac{n}{2}-1} dt \\ &= (2\pi)^{-(s+\frac{n}{2})} \Gamma(s + \frac{n}{2}) = L(\chi_{s,n}) \end{aligned}$$

where we recall our definition of the local L -factor in (2.1-2). A similar calculation also shows that $Z(f_n, \chi_{s,n}) =$

$L(\chi_{s,n})$ when $n < 0$.

It remains to compute the "dual" local zeta function. Classical Fourier duality tells us that we have

$$\widehat{f}_n(z) = i^{|n|} f_{-n}(z)$$

for all n . Next, we notice that $\chi_{n,s}^\vee = \chi_{1-s,-n}$ for any s and n . Using the fact that our computation was linear in the exponents s and n , we deduce that

$$Z(\widehat{f}_n, \chi_{s,n}^\vee) = i^{|n|} L(\chi_{s,n}^\vee).$$

As a consequence we get that:

$$\begin{aligned} \gamma(\chi_{s,n}, \psi, dx) &= i^{|n|} \frac{L(\chi_{s,n}^\vee)}{L(\chi_{s,n})} \\ \implies \epsilon(\chi_{s,n}, \psi, dx) &= i^{|n|}, \end{aligned} \tag{2.1-8}$$

where again division by $L(\chi_{s,n})$ is okay since $\Gamma_{\mathbb{C}}(s)$ is (globally) nonvanishing and well-defined on the strip $\Re(s) \in (0, 1)$. This completes the complex case.

To conclude our primary analysis on the Archimedean local fields, we note by looking at all the (nowhere-vanishing!) local epsilon factors computed in (2.1-6), (2.1-7) and (2.1-8), and their corresponding local L -factors that $\gamma(\chi, \psi, dx)$ is indeed meromorphic as a function of χ , and that the ϵ -factor has the desired properties. Though we showed this for a specific *choice* of ψ and dx , in §2.2 we will show that these hold for any ψ and dx .

Moreover, we exhibited, for each connected component of $\mathcal{X}(F^\times)$, a function $f \in \mathcal{S}(F)$ for which $\chi \mapsto Z(f, \chi)$ on the strip $0 < \Re(\chi) < 1$ on that component is visibly not identically zero, as required for the last part of the proof of part (b) of Theorem 2.1.6. $\square$

Before we proceed complete the proof of the final part of the theorem for the non-Archimedean case, we need to build a tool that will be vital in simplifying later expressions of the local zeta integrals.

DEFINITION 2.1.8. Given a locally compact topological ring R with additive character $\psi : R \rightarrow S^1$, multiplicative character $\chi : R^\times \rightarrow S^1$, and Haar measure $d^\times u$ on the unit group, we define the associated **Gauss sum** to be:

$$g(\chi, \psi) := \int_{R^\times} \chi(u) \psi(u) d^\times u.$$

REMARK. Gauss sums turn out to be a hugely convenient notation that show up in a host of different applications, from theta-functions to cyclotomic fields. In fact, Weil made use of these Gauss-sums as a key step in his diagonal variety point-counting argument to motivate the Weil Conjectures in his seminal 1949 paper [Wei49].

We will use this language and notation of conductors and Gauss sums to simplify the integrands of the local zeta functions via following key lemma.

LEMMA 2.1.9. Let χ be a multiplicative character on $\mathcal{O}_F^\times$ and ψ be an additive character, with conductors $\mathfrak{p}^n$ and $\mathfrak{p}^m$ respectively. Define $d^\times x := dx/|x|$, for dx a Haar measure on F , and let c denote the scaling factor of the multiplicative measure $d^\times u$, as discussed in (2.1-4). Then:

- (a) If $m < n$ then $g(\chi, \psi) = 0$.
- (b) If $m = n$ then $|g(\chi, \psi)|^2 = c \text{Vol}(\mathcal{O}_F, dx) \text{Vol}(1 + \mathfrak{p}^n, d^\times x)$.
- (c) If $m > n$ then $|g(\chi, \psi)|^2 = c \text{Vol}(\mathcal{O}_F, dx) (\text{Vol}(1 + \mathfrak{p}^n, d^\times x) - \text{Vol}(1 + \mathfrak{p}^{m-1}, d^\times x))$.

Proof. Case 1: $m < n$

We decompose $\mathcal{O}_F^\times$ into the disjoint union of (finitely many!) cosets of the unit subgroups $U_m := 1 + \mathfrak{p}^m$. Note for any $a, b \in \mathcal{O}_F$ we have

$$\psi(a(1 + \pi^m b)) = \psi(a)\psi(a\pi^m b) = \psi(a),$$

by definition of the conductor $\mathfrak{p}^m = \langle \pi^m \rangle$. Since $n \geq m$, we note χ is constant on cosets of U_m in $\mathcal{O}_F^\times$, the Gauss sum thus becomes:

$$g(\chi, \psi) = \sum_{\mathcal{O}_F^\times / U_m} \left(\psi(a)\chi(a) \int_{U_m} \chi(u) d^\times u \right).$$

However since $m < n$, the subgroup U_m is a strict superset of U_n , and so, by definition of the conductor, the character χ is non-trivial on U_m . Recall the integral of a nontrivial character over a compact subgroup is zero, which in turn makes the entire Gauss sum zero, as desired. This proves (a).

Case 2: $m \geq n$

Expanding out the expression and making use of Fubini, we get:

$$\begin{aligned} |g(\chi, \psi)|^2 &= g(\chi, \psi) \overline{g(\chi, \psi)} \\ &= \int_{\mathcal{O}_F^\times} \chi(x)\psi(x) d^\times x \cdot \overline{\int_{\mathcal{O}_F^\times} \chi(y)\psi(y) d^\times y} \\ &= \iint_{\mathcal{O}_F^\times} \chi(xy^{-1})\psi(x-y) d^\times x d^\times y. \end{aligned}$$

Making the substitution $xy^{-1} \mapsto z$ the above integral becomes

$$\int_{\mathcal{O}_F^\times} \chi(z)h(z) d^\times z,$$

where

$$h(z) := \int_{\mathcal{O}_F^\times} \psi(yz - y) d^\times y = c \int_{\mathcal{O}_F^\times} \psi(y(z-1)) dy.$$

Note we can switch the Haar measure in the second equality using Equation 2.1-4 and the fact that $|y| = 1$ for all $y \in \mathcal{O}_F^\times$. Making use of the fact that $\mathcal{O}_F$ is a local ring, we get:

$$h(z) = c \left(\int_{\mathcal{O}_F} \psi(y(z-1)) dy - \int_{\mathfrak{p}} \psi(y(z-1)) dy \right).$$

Note that if $v_{\mathfrak{p}}(z-1) < m-1$ then in the above integrands $y(z-1)$ takes values outside of the conductor of ψ , and thus the integrands are non-trivial characters. If $v_{\mathfrak{p}}(z-1) = m-1$ then the integrand is trivial over $\mathfrak{p}$, but not $\mathcal{O}_F$, and otherwise it is trivial over all of $\mathcal{O}_F$. These observations give us:

$$h(z) = \begin{cases} 0, & v_{\mathfrak{p}}(z-1) < m-1 \\ -c \frac{\text{Vol}(\mathcal{O}_F, dx)}{q}, & v_{\mathfrak{p}}(z-1) = m-1 \\ c \text{Vol}(\mathcal{O}_F, dx) \left(1 - \frac{1}{q}\right), & v_{\mathfrak{p}}(z-1) \geq m. \end{cases}$$

All that remains is to substitute these cases back into the integral $\int \chi(z)h(z) d^\times z$ to show the desired equalities

in part (b) and part (c), which follows from direct computation, as the reader may verify. $\square$

Now we have the necessary tools and results to prove the final part of Theorem 2.1.6 in the non-Archimedean case. We first make a quick observation to classify the multiplicative characters we are working with. Recall from our discussion in the prelude to Definition 2.1.3 that any such character of $F^\times$ is of the form:

$$\chi_{s,n} := \left(x \mapsto |x|^s \omega \left(\frac{x}{|x|} \right) \right),$$

where $x \mapsto x/|x|$ is some *choice* of homomorphic retraction (which we may take to be the literal normalization map) to $\mathcal{O}_F^\times \hookrightarrow F^\times$ and ω is a unitary character on $\mathcal{O}_F^\times$ with conductor $\mathfrak{p}^n$. Note the indices s, n do not completely determine the character $\chi_{s,n}$, as there is some choice of ω to be made, but this notation will end up sufficing for the computations ahead. Note also that in the unramified case (i.e $n = 0$), the unitary character ω is trivial. This fact will be used in the proof.

Proof of part (c) in Theorem 2.1.6 in the non-Archimedean case. Supposing our additive character ψ has conductor $\mathfrak{p}^m$, we define:

$$f(x) := \begin{cases} \psi(x), & x \in \mathfrak{p}^{m-n} \\ 0, & \text{otherwise.} \end{cases} \quad (2.1-9)$$

This is a Schwartz-Bruhat function as it is compactly-supported by definition, and also locally constant, since $\psi = \psi(0) = 1$ near 0, as $\mathbb{C}^\times$ has "no small subgroups". We now compute the local zeta integrals $Z(f, \chi_{s,n})$ in two cases: n zero, and n positive.

Case 1: $n = 0$

This means that ω is trivial, i.e $\chi_{s,0}$ is an unramified character. We recall ψ is trivial on $\mathfrak{p}^m$, and therefore f is zero outside of the conductor of ψ . Using similar methods as in the proof of Lemma 2.1.9, we compute:

$$\begin{aligned} Z(f, \chi_{s,0}) &= \int_{F^\times} f(x) \chi_{s,0}(x) d^\times x \\ &= \int_{\mathfrak{p}^m \setminus \{0\}} \psi(x) |x|^s \omega(x/|x|) d^\times x \\ &= \int_{\mathfrak{p}^m \setminus \{0\}} |x|^s d^\times x \\ &= \sum_{k \geq m} \left(\int_{\pi^k \mathcal{O}_F^\times} |x|^s d^\times x \right) \\ &= \text{Vol}(\mathcal{O}_F^\times, d^\times x) \sum_{k \geq m} q^{-ks} \\ &= q^{-ms} \text{Vol}(\mathcal{O}_F^\times, d^\times x) L(\chi_{s,0}), \end{aligned} \quad (2.1-10)$$

where we recall our definition of the local L -factor in the unramified case in (2.1-1).

Case 2: $n > 0$

Recalling that f is zero outside of $\mathfrak{p}^{m-n}$, by similar logic as in the $n = 0$ case we get:

$$Z(f, \chi_{s,n}) = \sum_{k \geq m-n} \left(\int_{\pi^k \mathcal{O}_F^\times} \psi(x) |x|^s \omega(x/|x|) d^\times x \right)$$

$$\begin{aligned}
&= \sum_{k \geq m-n} \left(\int_{\mathcal{O}_F^\times} \psi(\pi^k u) \left| \pi^k u \right|^s \omega(u) / \omega(\pi^k) d^\times u \right) \\
&= \sum_{k \geq m-n} q^{-ks} \left(\int_{\mathcal{O}_F^\times} \psi(\pi^k u) \omega(u) d^\times u \right) \\
&= \sum_{k \geq m-n} q^{-ks} g(\omega, \psi_{\pi^k}).
\end{aligned}$$

In the last line the Gauss sum notation comes as a typographical godsave for us, where $\psi_{\pi^k}(x) := \psi(x\pi^k)$. We note that the conductor of ψ_{π^k} is $m - k$, so in particular if $k > m - n$ then it contains the conductor of ω . We thus get:

$$Z(f, \chi_{s,n}) = q^{(n-m)s} g(\omega, \psi_{\pi^{m-n}}) = q^{(n-m)s} c \text{Vol}(\mathcal{O}_F, dx) \text{Vol}(1 + \mathfrak{p}^n, d^\times x) (\overline{g(\omega, \psi_{\pi^{m-n}})})^{-1}, \quad (2.1-11)$$

where the first equality follows from part (a) of Lemma 2.1.9 and the second from part (b) of the same lemma. Thus the zeta function, as a function of $s \in \mathbb{C}$ restricted to the strip $0 < \Re(s) < 1$, is a visibly holomorphic function with no zeroes or poles.

SANITY CHECK. Note that to define the inverse of the Gauss sum and to ensure $Z(f, \chi)$ is not identically zero in 2.1-11, we must have that $g(\omega, \psi_{m-n}) \neq 0$. But examining Lemma 2.1.9, since the conductor of ψ is $m - n < n$, (since $n \geq 0$), it follows that the Gauss sum is nonzero, so we may rest assured.

Now we need to compute the zeta function of the dual, where here we define the Fourier transform also with respect to our additive character ψ of conductor $\mathfrak{p}^m$, i.e.

$$\widehat{f}(y) := \int_F f(x) \psi(xy) dx = \int_{\mathfrak{p}^{m-n}} \psi(x(y+1)) dx. \quad (2.1-12)$$

We proceed with similar casework to before.

Case 1: $n = 0$

Since the conductor of ψ is $\mathfrak{p}^m$, we observe from (2.1-12) that $\widehat{f}(y)$ is the integral of $\psi(x(y+1))$ taken over $\mathfrak{p}^m$. The character is non-trivial on $\mathfrak{p}^m$, and thus $\widehat{f}(y)$ is zero, exactly when $y \notin \mathcal{O}_F$. Thus in this case we have:

$$\widehat{f}(y) = \begin{cases} \text{Vol}(\mathfrak{p}^m, dx), & y \in \mathcal{O}_F \\ 0, & \text{otherwise,} \end{cases}$$

i.e we have a scaled characteristic function of the ring of integers. Using this, we compute:

$$\begin{aligned}
Z(\widehat{f}, \chi_{s,0}^\vee) &= \int_{\mathcal{O}_F \setminus \{0\}} \widehat{f}(x) \chi_{s,0}^\vee(x) d^\times x \\
&= \text{Vol}(\mathfrak{p}^m, dx) \sum_{k \geq 0} \left(\int_{\mathcal{O}_F^\times} \left| \pi^k x \right|^{1-s} d^\times x \right) \\
&= \text{Vol}(\mathfrak{p}^m, dx) \text{Vol}(\mathcal{O}_F^\times, d^\times x) \frac{1}{1 - q^{(1-s)}} \\
&= \text{Vol}(\mathfrak{p}^m, dx) \text{Vol}(\mathcal{O}_F^\times, d^\times x) L(\chi_{s,0}^\vee). \quad (2.1-13)
\end{aligned}$$

Combing (2.1-10) and (2.1-13), we get that for $\chi = \chi_{s,0}$ we have:

$$\gamma(\chi, \psi, dx) = q^{ms} \text{Vol}(\mathfrak{p}^m, dx) \frac{L(\chi^\vee)}{L(\chi)}$$

$$\implies \epsilon(\chi, \psi, dx) = q^{ms} \text{Vol}(\mathfrak{p}^m, dx). \quad (2.1-14)$$

Case 2: $n > 0$

Examining (2.1-5) again, we see that the integrand $\psi(x(y+1)) = 1$ for all $x \in \mathfrak{p}^{m-n}$ exactly when $y+1 \in \mathfrak{p}^n$, and otherwise it is a nontrivial character of x and thus $\widehat{f}(y) = 0$. We deduce then that $\widehat{f}$ is the characteristic function of $\mathfrak{p}^n - 1$ scaled by the factor $\text{Vol}(\mathfrak{p}^{m-n}, dx)$. Noting that $\mathfrak{p}^n - 1 \subseteq \mathcal{O}_F^\times$ (since $n > 0$), we compute the dual local zeta function:

$$\begin{aligned} Z(\widehat{f}, \chi_{s,n}^\vee) &= \int_{\mathfrak{p}^n-1} \widehat{f}(x) \chi_{s,n}^\vee(x) d^\times x \\ &= \text{Vol}(\mathfrak{p}^{m-n}, dx) \int_{\mathfrak{p}^n-1} |x|^{1-s} \overline{\omega}(x) d^\times x \\ &= q^{n-m} \text{Vol}(\mathcal{O}_F, dx) \int_{1+\mathfrak{p}^n} \overline{\omega}(-1) \overline{\omega}(x) d^\times x \\ &= q^{n-m} \text{Vol}(\mathcal{O}_F, dx) \text{Vol}(1+\mathfrak{p}^n, d^\times x) \omega(-1), \end{aligned} \quad (2.1-15)$$

where the last line follows from the fact that the conductor of ω is equal to the conductor of its conjugate, so it is trivial on all of $1+\mathfrak{p}^n$. Combining (2.1-11) and (2.1-15) for $\chi = \chi_{s,n}$, we get:

$$\epsilon(\chi, \psi, dx) = \gamma(\chi, \psi, dx) = \frac{1}{c} q^{(m-n)(s-1)\omega(-1)} \overline{g(\omega, \psi_{\pi^{m-n}})} = \frac{1}{c} q^{(m-n)(s-1)} g(\overline{\omega}, \psi_{\pi^{m-n}}). \quad (2.1-16)$$

The final equality above made use of the following identity (the validity of which we leave the reader to verify) that, for any Gauss sum:

$$\overline{g(\chi, \psi)} = \chi(-1) g(\overline{\chi}, \psi).$$

We now conclude our analysis. The equality involving the epsilon factor and relevant local L -factors have been shown in (2.1-14) and (2.1-16), from all of which $\gamma(\chi, \psi, dx)$ is visibly meromorphic in χ , completing our proof. $\square$

Our work is not yet complete. We have implicitly shown the necessary conditions to allow the proofs of parts (b) and (c) to hold in our computations, in that we have demonstrated nontrivial zeta functions and holomorphicity of the ϵ -factors on the strip $\Re(\chi) \in (0, 1)$ on each connected component of $\mathcal{X}(F^\times)$ to allow the γ -factor to be meromorphically continued to the entire space. This will be discussed this more explicitly in the following subsection. There we also analyze dependence of the γ -factors on choice of additive character and Haar measure, thus allowing the results to hold outside of the specific choices made in our computations.

§2.2 The Epsilon and Gamma Factors (with many tables)

In the previous subsection we proved a functional equation of the zeta function on local fields, which involved the so-called gamma and epsilon factors, which were functions of our multiplicative character χ , our fixed additive character ψ , and our choice of Haar measure dx on our local field. In here we analyze the dependency of these factors on all three inputs, and prove some results about these ϵ -factors in general, and their connection to the so-called root number. First, for ease of reference here we summarize all our computations of the previous subsection.

For each connected component of $\mathcal{X}(F^\times)$, our standard function $f \in \mathcal{S}(F)$ was constructed so that $\chi \mapsto Z(f, \chi)$ on the strip $\Re(\chi) \in (0, 1)$ is a nonvanishing holomorphic function, as can be seen in the second column

	Multiplicative quasi-character χ	Additive character ψ	Haar measure dx	Standard function f
$F = \mathbb{R}$	$\chi = \cdot ^s$	$e^{-2\pi i x}$	Lebesgue measure	$e^{-\pi x^2}$
	$\chi = \text{sgn} \cdot ^s$			$x e^{-\pi x^2}$
$F = \mathbb{C}$	$\chi_{s,n} : z \mapsto z ^s \left(\frac{z}{ z }\right)^n, n \in \mathbb{Z}$	$e^{-2\pi i(z+\bar{z})}$	$2 dz \wedge d\bar{z} $	$\frac{1}{2\pi}(\bar{z})^n e^{-2\pi z\bar{z}}, n \geq 0$
				$\frac{1}{2\pi}(z)^{-n} e^{-2\pi z\bar{z}}, n < 0$
F non-Arch	$\chi_{s,n} : x \mapsto x ^s \omega\left(\frac{x}{ x }\right), n = \text{conductor}(\omega) \in \mathbb{Z}_{\geq 0}$	Variable of conductor $\mathfrak{p}^m$	Variable	$\mathbb{1}_{\mathfrak{p}^{m-n}}$

Table 2.2.A: The standard additive characters, measures and functions on $0 < \Re(\chi) < 1$ on each connected component of $\mathcal{X}(F^\times)$ that the zeta integrals and epsilon factors were computed with respect to.

of Table 2.2.B. Note that in the case of F being non-Archimedean and χ being unramified, the non-vanishing part rests on the non-vanishing of the Gauss sum, which always holds by Lemma 2.1.9, since the conductor of ψ_{m-n} is never less than n (the conductor of ω). This makes the proof of part (b) valid even in the case where $Z(f, \chi) = 0$.

We also record here the ϵ -factors corresponding each choice of χ, ψ and dx , all of which are visibly holomorphic as functions of s .

	Multiplicative quasi-character χ	Zeta function $Z(f, \chi)$	$\epsilon(\chi, \psi, dx)$
$F = \mathbb{R}$	$\chi = \cdot ^s$	$\Gamma_{\mathbb{R}}(s)$	1
	$\chi = \text{sgn} \cdot ^s$	$\Gamma_{\mathbb{R}}(s+1)$	i
$F = \mathbb{C}$	$\chi_{s,n}$	$\Gamma_{\mathbb{C}}\left(s + \frac{ n }{2}\right)$	$i^{ n }$
F non-Arch	$\chi_{s,n}, n = 0$	$\frac{q^{-ms}}{1-q^{-s}} \text{Vol}(\mathcal{O}_F^\times, d^\times x)$	$q^{ms} \text{Vol}(\mathfrak{p}^m, dx)$
	$\chi_{s,n}, n > 0$	$q^{(n-m)s} c \text{Vol}(\mathcal{O}_F, dx) \text{Vol}(1 + \mathfrak{p}^n, d^\times x) \overline{g(\omega, \psi_{\pi^{m-n}})}^{-1}$	$\frac{1}{c} q^{(m-n)(s-1)} g(\bar{\omega}, \psi_{\pi^{m-n}})$

Table 2.2.B: The zeta functions and epsilon factors with inputs from Table 2.2.A, which we omit from this table for sake of space.

Finally, we tabulate the local L -factors and gamma factors.

	Multiplicative quasi-character χ	Local $L(\chi)$	$\gamma(\chi, \psi, dx)$
$F = \mathbb{R}$	$\chi = \cdot ^s$	$\Gamma_{\mathbb{R}}(s)$	$\Gamma_{\mathbb{R}}(1-s)/\Gamma_{\mathbb{R}}(s)$
	$\chi = \text{sgn} \cdot ^s$	$\Gamma_{\mathbb{R}}(s+1)$	$i\Gamma_{\mathbb{R}}(1-s)/\Gamma_{\mathbb{R}}(s)$
$F = \mathbb{C}$	$\chi_{s,n}$	$\Gamma_{\mathbb{C}}\left(s + \frac{ n }{2}\right)$	$i^{ n } \Gamma_{\mathbb{C}}\left(1-s + \frac{ n }{2}\right) / \Gamma_{\mathbb{C}}\left(s + \frac{ n }{2}\right)$
F non-Arch	$\chi_{s,n}, n = 0$ (unramified)	$\left(1 - \pi_F ^{-s}\right)^{-1}$	$q^{ms} \text{Vol}(\mathfrak{p}^m, dx)^{\frac{1-\chi(\pi_F)}{1-\chi^v(\pi_F)}}$
	$\chi_{s,n}, n > 0$ (ramified)	1	$\frac{1}{c} q^{(m-n)(s-1)} g(\bar{\omega}, \psi_{\pi^{m-n}})$

Table 2.2.C: The local gamma factors with inputs from Table 2.2.A, again omitted for sake of space.

Table 2.2.C allows us to see at once that $\gamma(\chi, \psi, dx)$ is non-vanishing and holomorphic on the strip $\Re(\chi) \in (0, 1)$ for any representative χ of a connected component of $\mathcal{X}(F^\times)$, and thus it admits meromorphic continuation

to all of $\mathcal{X}(F^\times)$.

We will now see how the γ -factor behaves as ψ and dx vary. This will ensure our computations in the previous subsection easily imply the desired results for arbitrary choices of ψ and dx .

PROPOSITION 2.2.1. Let F be a local field, χ a nontrivial character of $F^\times$, ψ a character of F and dx a Haar measure. Then

(a) For every $t \in \mathbb{R}_{>0}$,

$$\gamma(\chi, \psi, t \cdot dx) = t \cdot \gamma(\chi, \psi, dx).$$

(b) For any $a \in F^\times$, if $\psi_a : x \mapsto \psi(ax)$ then

$$\gamma(\chi, \psi_a, dx) = \chi(a) |a|_F^{-1} \gamma(\chi, \psi, dx).$$

Proof. Since the Fourier transform of any $f \in \mathcal{S}(F)$ is dependent on choice of ψ, dx , we will write $\widehat{f}(y, \psi, dx)$ to denote this dependency (note if (ψ, dx) is not a self-dual pair then results like Fourier inversion need not hold, but this is no trouble for us presently). We thus have

$$\widehat{f}(y, \psi, t \cdot dx) := \int_F f(x) \psi(xy) t \, dx = t \widehat{f}(y, \psi, dx).$$

We also note that $Z(f, \chi)$ has a dependence on dx , if we define $d^\times x := dx / |x|_F$. In particular the scaling of $d^\times x$ is precisely the amount we scale dx by. It is also clear from definition that $Z(f, \chi, t \cdot dx) = t Z(f, \chi, dx)$. As we have seen, we may pick some $f \in \mathcal{S}(F)$ so that $Z(f, \chi, dx)$ is not identically 0 as a meromorphic function of χ on its connected component in $\mathcal{X}(F^\times)$, so we get:

$$\begin{aligned} \frac{Z(\widehat{f}(y, \psi, dx), \chi^\vee, t \cdot dx)}{Z(f, \chi, t \cdot dx)} &= \frac{t^2 Z(\widehat{f}(y, \psi, dx), \chi)}{t Z(f, \chi, dx)} \\ \implies \gamma(\chi, \psi, t \cdot dx) &= t \gamma(\chi, \psi, dx), \end{aligned}$$

thus proving part (a).

Similarly, we have

$$\widehat{f}(y, \psi_a, dx) := \int_F f(x) \psi(axy) \, dx = \widehat{f}(ay, \psi, dx).$$

Thus, we have

$$\begin{aligned} Z(\widehat{f}(x, \psi_a, dx), \chi^\vee) &= \int_{F^\times} \widehat{f}(ax, \psi, dx) \chi^\vee(x) d^\times x \\ &= \int_{F^\times} \widehat{f}(x, \psi, dx) \chi^\vee(a^{-1}x) d^\times x \\ &= \chi^\vee(a^{-1}) Z(\widehat{f}(x, \psi, dx), \chi^\vee) \\ &= \chi(a) |a|_F^{-1} Z(\widehat{f}(x, \psi, dx), \chi^\vee), \end{aligned}$$

from which it easily follows that

$$\gamma(\chi, \psi_a, dx) = \chi(a) |a|_F^{-1} \gamma(\chi, \psi, dx),$$

proving part (b). □

Since any possible Haar measure on F is obtained by scaling some chosen dx by some nonzero positive constant, and (as Proposition 1.2.4 tells us) any additive character is of the form ψ_a for some fixed ψ , the above result

generalizes our casework in the previous subsection completely. Thus we may (finally!) put our minds to rest on the validity of the proof of Theorem 2.1.6.

For the remainder of the subsections we sketch out some intermediary results that will come up later in §4 when we look at applications of Tate's thesis to explicit L -function computations.

In the case that the pair (ψ, dx) is self-dual, we also have nice results about the interaction of the γ -factors with the multiplicative character χ .

PROPOSITION 2.2.2. Let F be a local field, Suppose the pair (ψ, dx) is self-dual on F (i.e the Fourier inversion formula with respect to the additive character ψ and measure dx holds on F). Then the gamma factors satisfy:

- (a) $\gamma(\chi, \psi, dx) \gamma(\chi^\vee, \psi, dx) = \chi(-1)$.
- (b) $\gamma(\bar{\chi}, \psi, dx) = \chi(-1) \overline{\gamma(\chi, \psi, dx)}$.

Proof. Part (a): By Fourier inversion, classical Fourier theory that $\widehat{\widehat{f}}(x) = f(-x)$. Moreover, we make the observation that taking the shifted dual is an involution on the space of multiplicative characters, i.e. $(\chi^\vee)^\vee = \chi$. Putting these two facts together, we get:

$$Z(\widehat{\widehat{f}}, (\chi^\vee)^\vee) = \int_{F^\times} f(-x) \chi(x) d^\times x = \chi(-1) \int_{F^\times} f(-x) \chi(-x) d^\times x = \chi(-1) Z(f, \chi).$$

Next, recalling the main functional equation in Theorem 2.1.6 from the local theory and applying it twice, with the above equality, we get:

$$\begin{aligned} \chi(-1) Z(f, \chi) &= Z(\widehat{\widehat{f}}, (\chi^\vee)^\vee) \\ &= \gamma(\chi^\vee, \psi, dx) Z(\widehat{f}, \chi^\vee) \\ &= \gamma(\chi^\vee, \psi, dx) \gamma(\chi, \psi, dx) Z(f, \chi). \end{aligned}$$

For any connected component of $\mathcal{X}(F^\times)$ we may look in Table 2.2.A to find a function $f \in \mathcal{S}(F)$ such that Table 2.2.B will assure us that $\chi \mapsto Z(f, \chi)$ is not identically zero on the chosen connected component. Thus, we may cancel the $Z(f, \chi)$ without worry to obtain the desired equality.

Part (b): First, using the fact that $\psi(xy) = \overline{\psi(-xy)}$, since ψ is additive, for any $f \in \mathcal{S}(F)$ we have:

$$\begin{aligned} \widehat{\widehat{f}}(y) &:= \int_F \overline{\widehat{f}(x)} \psi(xy) dx \\ &= \int_F \overline{\widehat{f}(x)} \overline{\psi(-xy)} dx = \overline{\widehat{\widehat{f}}(-y)}. \end{aligned}$$

This gives:

$$\begin{aligned} Z(\widehat{\widehat{f}}, \chi) &:= \int_{F^\times} \widehat{\widehat{f}}(x) \chi(x) d^\times x \\ &= \int_{F^\times} \overline{\widehat{\widehat{f}}(-x)} \chi(x) d^\times x \\ &= \chi(-1) Z(\widehat{\widehat{f}}, \chi). \end{aligned}$$

Next we make the observation that complex conjugation commutes with taking the shifted dual, i.e. $\bar{\chi}^\vee = \overline{\chi^\vee}$. Applying this fact, along with the above deduced identity and the functional equation of Theorem 2.1.6, we get:

$$\begin{aligned} \overline{Z(\widehat{f}, \chi^\vee)} &= Z(\widehat{\bar{f}}, \overline{\chi^\vee}) = \chi(-1) Z(\widehat{f}, \bar{\chi}^\vee) \\ &= \chi(-1) \gamma(\bar{\chi}, \psi, dx) Z(\bar{f}, \bar{\chi}). \end{aligned} \quad (2.2-1)$$

However, we also have:

$$\overline{Z(\widehat{f}, \chi^\vee)} = \overline{\gamma(\chi, \psi, dx) Z(f, \chi)} = \overline{\gamma(\chi, \psi, dx)} Z(\bar{f}, \bar{\chi}). \quad (2.2-2)$$

Combining (2.2-1) and (2.2-2), and once again looking in our tables to pick f so that $\chi \mapsto Z(\bar{f}, \bar{\chi})$ is not identically zero on a chosen component of $\mathcal{X}(F^\times)$, we get desired equality. $\square$

COROLLARY 2.2.3. The statements of Propositions 2.2.1 and 2.2.2 hold true if we replace $\gamma(\chi, \psi, dx)$ with $\epsilon(\chi, \psi, dx)$.

Proof. This is immediate from the relationship between the epsilon factor and the gamma factor (as meromorphic functions of χ on $\mathcal{X}(F^\times)$) as defined in part (c) of Theorem 2.1.6, which we repeat below for convenience:

$$\epsilon(\chi, \psi, dx) = \gamma(\chi, \psi, dx) \frac{L(\chi)}{L(\chi^\vee)}.$$

Here we also implicitly use the observation (which the reader may choose to rigorously verify should they desire) that $\overline{L(\chi)} = L(\bar{\chi})$ for all the local L -factors and all inputs χ . $\square$

The following definition was motivated by several classical computations in analytic number theory, in which L -functions have certain complex factors of norm 1 attached to them known as root numbers. Our local zeta functions give us a nice characterizations of this factor.

DEFINITION 2.2.4. Let χ be a multiplicative character (i.e S^1 -valued) of a local field F , and let (ψ, dx) be a self-dual pair of an additive character and Haar measure on F . Then define the **root number** $W(\chi)$ by

$$W(\chi) := \epsilon(\chi \cdot |\cdot|^{1/2}, \psi, dx).$$

REMARK. At first glance this definition, in particular the choice of appending the factor of $|\cdot|^{1/2}$ to χ , may seem rather arbitrary. However, we will see that ensuring the input of the ϵ -factor is always a quasi-character of exponent 1/2 is crucial to make the root number well-defined.

COROLLARY 2.2.5. For any multiplicative character χ of a local field, the root number $W(\chi)$ is well-defined independent of choice of ψ and dx , and moreover $|W(\chi)| = 1$.

Proof. First we analyze dependence (or lack-thereof) on choice of additive character and Haar measure; let (ψ, dx) be a self-dual pair. By Theorem 1.2.4, any other additive character looks like $\psi_a = \psi(ax)$, for some $a \in F^\times$. One then knows (from classical Fourier theory in the Archimedean case and from examination of our proof of Theorem 1.2.4 in the non-Archimedean case) that $|a|^{1/2} dx$ is the Haar measure that gives rise to a self-dual pair with ψ_a .

Applying Proposition 2.2.1 (using Corollary 2.2.3 to use it for ϵ -factors), we get

$$\epsilon(\chi|\cdot|^{1/2}, \psi_a, |a|^{1/2} dx) = |a|^{1/2} (\chi(a) |a|^{1/2}) |a|^{-1} \epsilon(\chi|\cdot|^{1/2}, \psi, dx) = \epsilon(\chi|\cdot|^{1/2}, \psi, dx).$$

Thus $W(\chi)$ is well-defined.

For the second assertion, all this is really saying is that any quasi-character of exponent 1/2 has epsilon factor with norm 1. Note for an arbitrary quasi-character χ' we always have $\chi'(x)(\chi')^\vee(x) = |x|$. If we impose an exponent of 1/2 then we also have $\chi'(x)\overline{\chi'}(x) = |x|$. Combining these identities together and cancelling out $\chi'(x)$ we get $\overline{\chi'} = (\chi')^\vee$.

Using this fact and combining statements (a) and (b) of Corollary 2.2.3 (really Proposition 2.2.2), we obtain

$$\epsilon(\chi', \psi, dx)^{-1} = \overline{\epsilon(\chi', \psi, dx)},$$

and thus the epsilon factor of any exponent 1/2 quasi-character has norm 1, proving our desired result. $\square$

SECTION 3

THE GLOBAL THEORY

§3.1 Adelic Fourier Theory (with some Riemann-Roch)

We first explore the theory of integration and Fourier transforms on general restricted products. Essentially we just want to confirm that all the results and identities one expects to hold actually do. Continuing with the notation of §1.1, we let $\{G_v\}$ be a collection of locally compact Hausdorff groups, cofinitely many of which are equipped with a respective compact subgroup H_v , a Haar measure dg_v with the property that $\text{Vol}(H_v, dg_v) = 1$ for almost all v , and let G denote their restricted product. We use dg to denote the induced Haar measure on the restricted product, as in Definition 1.2.13.

OBSERVATION. For f an integrable function on G , we have the following identity:

$$\int_G f(g) dg = \lim_S \int_{G_S} f(g_S) dg_S,$$

where the limit is naturally taken over an increasing directed union of places. This is a direct consequence of the fact that we may compute and integral via a limit of any compact exhaustion, and the fact that any compact is contained in some G_S . Conversely, the dominated convergence theorem tells us that if this limit exists for $|f|$ (for a Borel function f), then f must be integrable. In particular $f \in L^1(G)$ if and only if $|f| \in L^1(G)$.

Let S_0 be a finite set of indices containing J_∞ and the (finitely many!) finite v for which $\text{Vol}(H_v, dg_v) \neq 1$. Suppose for each v we have a continuous integrable f_v on G_v such that $f_v|_{H_v} = 1$ for all $v \notin S_0$. For $g = (g_v) \in G$, define

$$f(g) = \prod_v f_v(g_v)$$

PROPOSITION 3.1.1. With the notation above, the function f is well-defined and continuous. For any $S \supseteq S_0$, we have

$$\int_{G_S} f(g_S) dg_S = \prod_{v \in S} \int_{G_v} f_v(g_v) dg_v$$

Proof. Since $g = (g_v)$ is in the restricted direct product, then $g_v \in H_v$ for all but finitely many v . Combining this with the fact that $f_v|_{H_v} = 1$ for almost all v , we deduce $\prod f_v(g_v)$ is a finite product, and thus well-defined.

To show f is continuous, consider a base for G consisting of sets of the form

$$\prod U_v \times \prod H_v$$

where the first product contains all the (finitely many!) finite v such that $f|_{H_v}$ is nontrivial. Thus f can be computed locally as a finite product of continuous functions, and is thus locally continuous, and thus continuous.

The fact that the integral behaves well with products is a consequence of definitions and Fubini's theorem. $\square$

COROLLARY 3.1.2. With the notation as in the above result, if f is integrable then

$$\int_G f(g) \, dg = \prod_v \int_{G_v} f_v(g_v) \, dg_v$$

and if the right side exists for $|f|$ then f is integrable.

Proof. By our earlier observation, we know f is integrable if and only if $\lim_S \int_{G_S} |f|(g_S) \, dg_S$ exists, which is true if and only if the limit

$$\lim_S \prod_{v \in S} \int_{G_v} |f_v|(g_v) \, dg_v = \prod_v \int_{G_v} |f_v|(g_v) \, dg_v < \infty,$$

exists, by Proposition 3.1.1. □

PROPOSITION 3.1.3. Suppose all the G_v are abelian, and suppose the continuous functions $\{f_v\}$ on G_v are the characteristic functions for H_v for all but finitely many v . Then f is integrable, with Fourier transform $\widehat{f}$ continuous and given by

$$\widehat{f}(\chi) = \prod_v \widehat{f}_v(\chi_v),$$

for $\chi = (\chi_v)_v \in \widehat{G} = \prod_v \widehat{G}_v$.

Proof. Firstly note that we have

$$\widehat{f}(\chi) = \int_G f(g) \chi(g) \, dg = \prod_v \int_{G_v} f_v(g_v) \chi_v(g_v) \, dg_v = \prod_{v \in S} \int_{G_v} f_v(g_v) \chi_v(g_v) \, dg_v = \prod_{v \in S} \widehat{f}_v(\chi_v)$$

for some finite set S , since f_v is the characteristic function of H_v and $\chi_v|_{H_v} = 1$ for cofinitely many v . Thus we have a finite product of continuous functions of χ , as desired. □

From now on assume all groups are abelian. We want to build a Haar measure on the Pontryagin dual $\widehat{G}$ that is dual to dg in the sense of the Fourier inversion theorem. For each v , let

$$d\chi_v = \widehat{dg_v}$$

denote the dual measure to dg_v on $\widehat{G}_v$. Note for any $f_v \in L^1(G_v)$, we have that

$$\widehat{f}_v(\chi_v) = \int_{G_v} f_v(g_v) \overline{\chi_v}(g_v) \, dg_v$$

by definition. If f_v is the characteristic function of H_v (which is visibly integrable and nonnegative) then

$$\widehat{f}_v(\chi_v) = \int_{H_v} \chi_v(g_v) \, dg_v = \begin{cases} \text{Vol}(H_v, dg_v), & \chi_v|_{H_v} = 1 \\ 0, & \text{otherwise,} \end{cases}$$

since the integral of a non-trivial character over a compact group is zero. In other words, if $H_v^\perp$ is the subgroup of $\widehat{G}_v$ of characters trivial on H_v , then $\widehat{f}_v(\chi_v)$ is the characteristic function of $H_v^\perp$ scaled by $\text{Vol}(H_v, dg_v)$. Thus, by Fourier inversion, for $y \in H_v$ we have:

$$\begin{aligned} \widehat{f}_v(y) &= \int_{\widehat{G}_v} \widehat{f}_v(\chi_v) \chi_v(y) \, d\chi_v \\ &= \text{Vol}(H_v, dg_v) \int_{H_v^\perp} d\chi_v \end{aligned}$$

$$= \text{Vol}(H_v, dg_v) \text{Vol}(H_v^\perp, d\chi_v)$$

where the first equality follows from our choice of f_v as the characteristic function of H_v . Recall in our construction of dg that dg_v assigns volume 1 to H_v for all but finitely many v . The above equation then tells us $\text{Vol}(H_v^\perp, d\chi_v) = 1$ almost everywhere as well, and so we can define the restricted product Haar measure

$$d\chi := \prod_v d\chi_v \quad (3.1-1)$$

as in (1.2-5), and this is also well-defined.

PROPOSITION 3.1.4. The measure $d\chi$, as defined in (3.1-1), is dual to dg , defined in (1.2-5), in the sense that:

$$f(g) = \int_{\widehat{G}} \widehat{f}(\chi) \chi(g^{-1}) d\chi$$

for all continuous $f \in L^1(G)$ such that $\widehat{f} \in L^1(\widehat{G})$.

Proof. For our purposes, since we know a dual measure to dg exists and is unique, it suffices to check on some nonzero continuous $f \in L^1(G)$, which we may choose to be a product function of the form

$$f = \prod f_v,$$

where f_v is the characteristic function of H_v for almost all v . For any input g , the left side above is a locally finite product $\prod f_v(g_v)$ by definition, and using the fact that each $d\chi_v$ is dual to dg_v , we get:

$$f_v(g_v) = \int_{\widehat{G}_v} \widehat{f}_v(\chi_v) \chi_v(g_v^{-1}) d\chi_v.$$

Multiplying them together (while applying Fubini) gives us our desired result. $\square$

With the relevant desired properties verified, we are now ready to proceed to some actual adelic harmonic analysis.

DEFINITION 3.1.5. For K a global field, if for every place v we have some $f_v \in \mathcal{S}(K_v)$ such that $f_v|_{\mathcal{O}_v} = 1$ for almost every v , then the "tensor product"

$$\bigotimes_v f_v =: f : \mathbb{A}_K \rightarrow \mathbb{C}$$

$$(x_v) \mapsto \prod f_v(x_v)$$

is locally a finite product, and we call any $\mathbb{C}$ -linear combination of such functions an **adelic Schwartz-Bruhat function**. We denote by $\mathcal{S}(\mathbb{A}_K)$ the complex vector space of all such functions.

Similar to classical Fourier analysis, it is an exercise in simple-function approximation to see that $\mathcal{S}(\mathbb{A}_K)$ is dense in $L^2(\mathbb{A}_K)$ and $L^1(\mathbb{A}_K)$. Our goal will be to derive an adelic version of the Poisson summation formula of classical Fourier analysis.

First off, to define our Fourier transform we fix ψ to be a standard additive character on the adèles, i.e any character trivial on $K \hookrightarrow \mathbb{A}_K$. Moreover we let dx be the (self-dual!) Tamagawa measure on $\mathbb{A}_K$, defined in

1.2.15. Then for any $f \in \mathcal{S}(\mathbb{A}_K)$, the Fourier transform

$$\widehat{f}(x) := \int_{\mathbb{A}_K} f(y) \psi(xy) dy \quad (3.1-2)$$

makes sense.

PROPOSITION 3.1.6. The Fourier transform $f \mapsto \widehat{f}$ is an L^2 -isometry of $\mathcal{S}(\mathbb{A}_K)$.

Proof. This follows immediately from the fact that on each component it is an isometry (from classical Fourier analysis at the Archimedean places, and from our proof of Theorem 1.2.4 at the non-Archimedean places), and from the multiplicative definition of the idelic norm in Definition 1.3.5. We leave it to the reader to verify the (rather tedious) details should they want to do so. $\square$

We will also particularly be interested in continuous integrable functions on $\mathbb{A}_K$ that are K -periodic, i.e. $f(x) = f(x + \alpha)$ for every $\alpha \in K$. In these cases we may view f as a function on the quotient $\mathbb{A}_K/K$. In particular, we may consider its Fourier transform on $\widehat{\mathbb{A}_K/K}$ (canonically identified with K by Proposition 1.2.10, up to some global choice of nonzero $\omega \in \Omega_{K/\mathbb{F}_q}^1$ in the function field case)

$$\widehat{f}(\alpha) := \frac{1}{\text{Vol}(\mathbb{A}_K/K, dx)} \int_{\mathbb{A}_K/K} f(y) \psi(\alpha y) dy, \quad (3.1-3)$$

for $\alpha \in K$.

WARNING (Notational Abuse). There are a few slight abuses of notation here; we have already mentioned the one with our notation of f . We also abuse notation for the measure dy in (3.1-3) is unique measure induced on the quotient by the self-dual Tamagawa dx on $\mathbb{A}_K$ and the counting measure on K . We also multiply by $1/(\text{Vol}(\mathbb{A}_K/K))$ since this is the normalized measure on the quotient that is dual of the counting measure on K . Theorem 1.2.16 tells us this volume is 1, but we will actually deduce this as a consequence of the Poisson summation formula.

Fourier inversion then tells us that if $\widehat{f} \in L^1(K)$ then

$$f(x) = \sum_{\alpha \in K} \widehat{f}(\alpha) \overline{\psi}(x\alpha)$$

for $x \in \mathbb{A}_K/K$, since a convergent integral with respect to a counting measure is just an absolutely convergent sum.

DEFINITION 3.1.7. A $\mathbb{C}$ -valued continuous integrable function f on $\mathbb{A}_K$ is **admissible** if both

$$\sum_{\gamma \in K} f(x + \gamma), \quad \sum_{\gamma \in K} \widehat{f}(x + \gamma)$$

are absolutely and uniformly convergent on compact sets.

Admissibility will be a vital property for us, as we will need to use it in the proofs of the upcoming results of Poisson Summation and Riemann-Roch for the adeles, so we had better be sure everything we are working with is indeed admissible.

LEMMA 3.1.8. Any $f \in \mathcal{S}(\mathbb{A}_K)$ is admissible.

Proof. Since the Fourier transform $f \mapsto \hat{f}$ carries $\mathcal{S}(\mathbb{A}_K)$ into itself, it suffices to just show the necessary convergence for an arbitrary adelic Schwartz-Bruhat function. By part (b) of Proposition 1.1.3, any compact subset of $\mathbb{A}_K$ is contained in some set of the form

$$C := \prod_{v \in S} C_v \times \prod_{v \notin S} \mathcal{O}_v$$

for S a finite index set containing the infinite places and $C_v \subseteq K_v$ compact. Thus it suffices to show (absolute and uniform) convergence on a set of the form C . Recall that, as a $\mathbb{C}$ -vector space, $\mathcal{S}(\mathbb{A}_K)$ is generated by product functions built from characteristic functions of fractional ideals at all the non-Archimedean places v , and some Schwartz function at all the Archimedean completions. Showing convergence of any such generating function suffices, as it will imply convergence of any finite $\mathbb{C}$ -linear combination of them.

For any such "product function" f we note that at any non-Archimedean place v there exists some $\delta_v \in \mathbb{R}_{>0}$ such that

$$|\gamma|_v > \delta_v \implies f_v(x_v + \gamma) = 0 \quad \forall x_v \in C_v.$$

We can use $\delta_v = 1$ for almost all v , and so there is some maximum δ_v across all finite places.

In the positive characteristic case, since our K is some function field over a finite field and every place is non-Archimedean, this tells us the sum $\sum_{\gamma \in K} f(x + \gamma)$ is just a finite sum on C , and thus naturally absolutely and uniformly convergent.

In the number field case, our summation is restricted to some fractional ideal J of $\mathcal{O}_K$ whose prime factorization corresponds exactly to all the finite places where $\delta_v \neq 1$. However note that this is a discrete subset of

$$K_\infty := \prod_{v|\infty} K_v,$$

which we can treat as a subset of a lattice in $\mathbb{R}^n$. Note that, as we take an exhaustion of closed balls, the number of elements in the lattice contained in a shell of radius R and thickness Δ (relatively small compared to R) grows like a fixed polynomial in R . However, since $f_v \in \mathcal{S}(K_v)$ decreases faster than any polynomial, it will go to 0 much faster than the number of γ in the sum can grow. This gives us our desired result. $\square$

Now that we know our summation does give us a well-defined value, the statement of the following theorem makes sense.

THEOREM 3.1.9 (Adelic Poisson Summation). For $f \in \mathcal{S}(\mathbb{A}_K)$,

$$\sum_{\gamma \in K} f(\gamma) = \frac{1}{\text{Vol}(\mathbb{A}_K/K, dx)} \sum_{\gamma \in K} \hat{f}(\gamma)$$

Proof. For notational convenience, define

$$F(x) := \sum_{\gamma \in K} f(x + \gamma).$$

Note that since $F(x)$ is K -periodic we can view it as a continuous function on $\mathbb{A}_K/K$. By Pontryagin duality considerations we then know its Fourier transform $\hat{F}$ can be viewed as a function $K \rightarrow \mathbb{C}$. For any $\alpha \in K$, we

have:

$$\begin{aligned}
\widehat{F}(\alpha) &= \frac{1}{\text{Vol}(\mathbb{A}_K/K, dx)} \int_{\mathbb{A}_K/K} \sum_{\gamma \in K} f(x + \gamma) \psi(\alpha x) dx \\
&= \frac{1}{\text{Vol}(\mathbb{A}_K/K, dx)} \int_{\mathbb{A}_K/K} \sum_{\gamma \in K} f(x + \gamma) \psi(\alpha(x + \gamma)) dx \\
&= \frac{1}{\text{Vol}(\mathbb{A}_K/K, dx)} \int_{\mathbb{A}_K} f(x) \psi(\alpha x) dx \\
&= \frac{1}{\text{Vol}(\mathbb{A}_K/K, dx)} \widehat{f}(\alpha),
\end{aligned}$$

where the second equality follows from the fact that ψ is trivial on K and the third follows from the compatibility of our measure on $\mathbb{A}_K/K$ with the counting measure on K . Since $\widehat{f} \in \mathcal{S}(\mathbb{A}_K)$, the above equality also tells us that $\widehat{F} \in L^1(K)$, so we may apply Fourier inversion to get

$$\begin{aligned}
F(x) &:= \sum_{\gamma \in K} f(x + \gamma) = \sum_{\gamma \in K} \widehat{F}(\gamma) \overline{\psi(x\gamma)} \\
&= \frac{1}{\text{Vol}(\mathbb{A}_K/K, dx)} \sum_{\gamma \in K} \widehat{f}(\gamma) \overline{\psi(x\gamma)}.
\end{aligned}$$

Setting $x = 0$ gives us our desired result. □

This result allows us to give a computation-free proof of the fact that the Tamagawa measure assigns the quotient $\mathbb{A}_K/K$ volume 1.

COROLLARY 3.1.10. The volume of $\mathbb{A}_K/K$, induced by the Tamagawa measure on $\mathbb{A}_K$ and the counting measure on K , is 1.

Proof. By Theorem 3.1.9, we have

$$\sum_{\gamma \in K} \widehat{f}(\gamma) = \text{Vol}(\mathbb{A}_K/K, dx) \sum_{\gamma \in K} f(\gamma).$$

Applying the Fourier transform again to $\widehat{f}$ and using Fourier inversion, we have:

$$\begin{aligned}
\sum_{\gamma \in K} f(\gamma) &= \sum_{\gamma \in K} f(-\gamma) \\
&= \sum_{\gamma \in K} \widehat{\widehat{f}}(\gamma) \\
&= \text{Vol}(\mathbb{A}_K/K, dx) \sum_{\gamma \in K} \widehat{f}(\gamma) \\
&= \text{Vol}(\mathbb{A}_K/K, dx)^2 \sum_{\gamma \in K} f(\gamma).
\end{aligned}$$

If we can find some $f \in \mathcal{S}(\mathbb{A}_K)$ such that $\sum_{\gamma \in K} f(\gamma) \neq 0$, the above expression would then imply $\text{Vol}(\mathbb{A}_K/K, dx)^2 = 1$, so the volume must in fact be equal to 1! It remains to construct such a Schwartz-Bruhat function.

When K is a global function field, we may simply take the adelic "tensor product" of the characteristic function of the local valuation rings at all places, i.e

$$f := \bigotimes_{v \in \mathcal{P}_K} \mathbb{1}_{\mathcal{O}_v}.$$

To be integral at all places a rational function $\gamma \in K$ must have no poles and come from a global function on the corresponding curve over $\mathbb{F}_q$; this is satisfied only by constant functions, so we have

$$\sum_{\gamma \in K} f(\gamma) = |\mathbb{F}_q| = q \neq 0,$$

as needed.

When K is a number field, at all the infinite places we may set f_v to be a non-negative nonzero bump function compactly supported in some neighborhood of 0, and at all the finite places we may set $f_v = \mathbb{1}_{\mathcal{O}_v}$, and set f to be the adelic "tensor" product of them. Since $\cap_{v < \infty} \mathcal{O}_v = \mathcal{O}_K$, it follows that the sum

$$\sum_{\gamma \in K} f(\gamma)$$

is nonzero and positive at some finite number of elements of $\mathcal{O}_K$, specifically the ones that lie in the support of f_v under all the embeddings (up to complex conjugation) $K \hookrightarrow \mathbb{C}$. By construction this includes $0 \in \mathcal{O}_K$, so in particular this sum is nonzero. This completes our proof. $\square$

We will also need a multiplicative version of the Poisson summation formula, relating to the average

$$\sum_{\gamma \in K} f(x\gamma),$$

for $x \in \mathbb{I}_K$. This will be analogous to the Riemann-Roch theorem in a way that we will discuss later in this subsection.

THEOREM 3.1.11 (Adelic Riemann-Roch). For any $x \in \mathbb{I}_K$ and $f \in \mathcal{S}(\mathbb{A}_K)$,

$$\sum_{\gamma \in K} f(\gamma x) = \frac{1}{|x|_{\mathbb{I}_K}} \sum_{\gamma \in K} \widehat{f}(\gamma x^{-1})$$

Proof. Fix some $x \in \mathbb{I}_K$, and set $g(z) := f(xz)$, for any $z \in \mathbb{A}_K$. Note that $g \in \mathcal{S}(\mathbb{A}_K)$ as well, and thus Poisson summation (3.1.9) applies to give us:

$$\begin{aligned} \sum_{\gamma \in K} g(\gamma) &= \sum_{\gamma \in K} \widehat{g}(\gamma) \\ &= \sum_{\gamma \in K} \int_{\mathbb{A}_K} f(xz) \psi(\gamma z) \, dz \\ &= \sum_{\gamma \in K} \int_{\mathbb{A}_K} \frac{1}{|x|_{\mathbb{I}_K}} f(z) \psi(\gamma z x^{-1}) \, dz \\ &= \frac{1}{|x|_{\mathbb{I}_K}} \sum_{\gamma \in K} \widehat{f}(\gamma x^{-1}), \end{aligned}$$

completing the proof. $\square$

At first glance, one might wonder why the above Theorem is named Riemann-Roch; indeed, there seems to be no compact Riemann surfaces anywhere in the picture. Even in Tate's exposition he does not expound upon this connection apart from declaring that it is a "number-theoretic analogue" of Riemann-Roch. In general, the theory of local and global fields in positive characteristic cannot be fully fleshed out and understood without the language of algebraic geometry, and to that end we indulge in a short interlude for the remainder of this subsection to explain this connection. Due the limits of space and time however we will not be able to describe nearly as many

connections to broader concepts as we should; for a more detailed exposition we refer the reader to the second chapter of [Ser88], or the more number-theory-oriented (yet denser) sixth chapter of [Wei74].

The goal here will be to view K as the function field of a smooth proper curve X that is geometrically connected over the finite field of constants $\mathbb{F}_q \subseteq K$, and reformulate Theorem 3.1.11 in geometric terms that will correspond to the classical theorem of Riemann-Roch.

DEFINITION 3.1.12. Let K be a global field of positive characteristic. A **divisor** on K is a finite formal linear combination

$$D = \sum_{v \in \mathcal{P}_K} n_v \cdot v, \quad n_v \in \mathbb{Z}.$$

We denote by $\text{Div}(K)$ the group of divisors on K (which can be thought of as the free abelian group generated by the places of K).

The **degree** of a place $\deg(v)$ is the degree of the residue field corresponding to v over the field of constants $\mathbb{F}_q$. For a divisor D we define its degree by

$$\deg(D) := \sum_{v \in \mathcal{P}_K} n_v \deg(v).$$

We denote by $\text{Div}^0(K)$ the subgroup of degree-zero divisors.

For any $f \in K^\times$, we note the formal linear combination

$$\text{div}(f) := \sum_{v \in \mathcal{P}_K} \text{ord}_v(f) v \tag{3.1-4}$$

is a finite linear combination, since f and $1/f$ are both regular functions on a dense open in X . Hence this is a divisor, and moreover we note that

$$\text{div}(f) + \text{div}(g) = \text{div}(fg)$$

Furthermore, recall the product formula (1.3.6) tells us for any $f \in K^\times$ that

$$\begin{aligned} 1 &= \prod_{v \in \mathcal{P}_K} |f|_v = \prod_{v \in \mathcal{P}_K} q_v^{\text{ord}_v(f)} \\ &= q^{\sum_{v \in \mathcal{P}_K} \deg(v) \text{ord}_v(f)} \\ \implies \deg(\text{div}(f)) &= \sum_{v \in \mathcal{P}_K} \deg(v) \text{ord}_v(f) = 0. \end{aligned}$$

Thus any such divisor induced by an element of $K^\times$ lies in $\text{Div}^0(K)$, which motivates the following definitions.

DEFINITION 3.1.13. We denote by $\text{div}(K^\times)$ the group of all **principal divisors** of the form $\text{div}(f)$ for $f \in K^\times$. The quotient $\text{Div}(K) / \text{div}(K^\times)$ is denoted $\text{Pic}(K)$ and called the **Picard group** of K , whose elements are called **divisor classes**. We also define the **Picard group of degree zero**, denoted $\text{Pic}^0(K)$, by the quotient $\text{Div}^0(K) / \text{div}(K^\times)$.

We note that if $\text{div}(f) = \text{div}(g)$ then $\text{div}(f/g) = 0$, which means $f/g \in \bigcap_{v \in \mathcal{P}_K} \mathcal{O}_v \implies f/g \in \mathbb{F}_q^\times$. All

of this tells us we have the following exact sequence:

$$1 \rightarrow \mathbb{F}_q^\times \rightarrow K^\times \rightarrow \operatorname{Div}^0(K) \rightarrow \operatorname{Pic}^0(K) \rightarrow 0.$$

DEFINITION 3.1.14. We say a divisor $D = \sum_{v \in \mathcal{P}_K} n_v v$ is **effective** if $n_v \geq 0$ for all v .

The **linear system** $L(D)$ of a divisor D is the union of $\{0\}$ and set of all $f \in K^\times$ such that $D + \operatorname{div}(f)$ is effective.

Geometrically, the notion of an effective divisor corresponds to some (codimension-one) subvariety cut out by regular functions, not just rational ones. Analytically one could also think of it as the vanishing locus of a holomorphic function. This concept also allows us to "compare" divisors to one another. In particular, we say

$$D \geq D' \iff D - D' \text{ is effective} \iff n_v \geq n'_v \forall v \in \mathcal{P}_K.$$

OBSERVATION. The "functions" $f \in K^\times$ that are in $L(D)$, together with 0, form an $\mathbb{F}_q$ -vector space, since $f \in L(D)$ if and only if $\operatorname{ord}_v(f) \geq -n_v$ for all $v \in \mathcal{P}_K$. We make the observation that if $\deg(D) < 0$ then $\dim(L(D)) = 0$. In the more general case, we have the following proposition.

PROPOSITION 3.1.15. For any divisor D , the $\mathbb{F}_q$ -vector space $L(D)$ is finite-dimensional.

Proof. The proof here uses the idelic formulation, which will be used to relate Theorem 3.1.11 to the Riemann-Roch theorem on X . First we define a *divisor map* on $\mathbb{I}_K$ by

$$\begin{aligned} \mathcal{D} : \mathbb{I}_K &\rightarrow \operatorname{Div}(K) \\ (x_v)_v &\mapsto \sum_v \operatorname{ord}_v(x_v) \cdot v. \end{aligned} \tag{3.1-5}$$

This map is visibly a surjective homomorphism, so to each divisor $D \in \operatorname{Div}(K)$ we may denote an idele $x(D) = (x(D)_v)_v$, i.e. $\operatorname{ord}_v(x(D)_v) = \operatorname{ord}_v(D)$. Letting f_v be the characteristic function of $\mathcal{O}_v$ for each v , we may define $f := \bigotimes_v f_v \in \mathcal{S}(\mathbb{A}_K)$. Then by construction, for any $\gamma \in K^\times$, we have:

$$f(\gamma x(D)) = \begin{cases} 1, & \text{if } \operatorname{ord}_v(\gamma x(D)_v) \geq 0 \forall v, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $f(\gamma x(D)) = 1$ if and only if $\mathcal{D}(\gamma x(D))$ is an effective divisor, which in turn is true if and only if $\gamma \in L(D)$. Thus

$$\sum_{\gamma \in K} f(\gamma x(D)) = |L(D)|.$$

Note f is admissible by construction and so the adelic Riemann-Roch theorem applies to tell us the above sum converges, and thus $L(D)$ has finite cardinality, hence is finite dimensional. $\square$

REMARK. The proof above is rather non-standard, for example a more classical approach taken in Chapter II of [Ser88] was to show that the sheaf $\mathcal{L}(D)$ – defined by having its non-zero sections over any nonempty open $U \subseteq X$ consist of regular functions satisfying $\operatorname{div}_U(f) \geq D \cap U$ – is coherent. Then one can use the well-known result that the global sections of any coherent sheaf on a projective variety are finite-dimensional.

However, our proof method is important for two reasons: (1) as mentioned, it will be used later in the proof of the Riemann-Roch theorem, and (2) it demonstrates the versatility of the adelic language.

It will also be of later use to us to note some basic properties of the divisor map, all of which follow directly from definitions. We leave the reader to verify the following equalities, should they wish.

PROPOSITION 3.1.16. If $\mathcal{D} : \mathbb{I}_K \rightarrow \text{Div}(K)$ denotes the divisor map, then the following properties hold:

- $\ker(\mathcal{D}) = \prod_v \mathcal{O}_v^\times = I_{K,\emptyset}$.
- $\mathcal{D}(\mathbb{I}_K^1) = \text{Div}^0(K)$.
- $\mathbb{I}_K / (K^\times \cdot \mathbb{I}_{K,\emptyset}) = \text{Pic}(K)$.
- $\mathbb{I}_K^1 / (K^\times \cdot \mathbb{I}_{K,\emptyset}) = \text{Pic}^0(K)$.

REMARK. The nontrivial map $\deg : \text{Pic}(K) \rightarrow \mathbb{Z}$ not only has finite index, but is in fact *surjective*. This arises as a consequence of the fact that the greatest common divisor of $\deg(v)$ over all $v \in \mathcal{P}_K$ is 1. This is discussed further in the proof of Proposition 4.1.3 and the remark following it.

We are now ready for the formulation of the Riemann-Roch theorem.

THEOREM 3.1.17 (Geometric Riemann-Roch). Let K be a global field of positive characteristic. Then there exists an integer $g \geq 0$, called the **genus** of K , and a divisor $\mathcal{K}$ of degree $2g - 2$, called the **canonical divisor** of K , such that

$$\dim(L(D)) - \dim(L(\mathcal{K} - D)) = \deg(D) - g + 1$$

for all $D \in \text{Div}(K)$.

The proof of this theorem relies on the extension of the notion of a divisor to that of characters as well. Let ψ be a non-trivial character of $\mathbb{A}_K$ trivial on K ; we know this exists by Proposition 1.2.8. Moreover, the Breaking Down Lemma (1.2.5) tells us that the induced characters ψ_v on each are trivial on $\mathcal{O}_v$ at cofinitely many v . Thus, if we define $\text{ord}_v(\psi_v) := \text{ord}_{\mathfrak{p}_v}(\text{Conductor of } \psi_v)$, we get that the divisor

$$\text{div}(\psi) := - \sum_v \text{ord}_v(\psi_v) \cdot v \tag{3.1-6}$$

is well-defined. Fixing any such ψ , Proposition 1.2.10 tells us that any other such character of $\mathbb{A}_K$ trivial on K is obtained by $x \mapsto \psi(\alpha x)$ for some $\alpha \in K^\times \subseteq \mathbb{I}_K$. In other words, the set of all divisors induced by characters of $\mathbb{A}_K/K$ is given by a single divisor class modulo the group of principal divisors, i.e some element $\Psi \in \text{Pic}(K)$. This is the key to the proof.

Proof of 3.1.17. Pick any $\mathcal{K} \in \Psi$ defined above corresponding to a standard character ψ of $\mathbb{A}_K$ trivial on K , constructed in the proof of Proposition 1.2.8 via a choice of nonzero global $\omega \in \Omega_{K/\mathbb{F}_q}^1$. As seen in that proof, each local ψ_v has conductor $\mathfrak{p}_v^{-d_v}$, where d_v is the order of vanishing of ω in at the stalk $\Omega_{X,v}^1$. By construction, this conductor is equal to $\mathfrak{p}_v^{\deg(v) \text{ord}_v(\mathcal{K})}$.

Now define

$$g := 1 + \frac{\deg(\mathcal{K})}{2} \in \frac{1}{2}\mathbb{Z}.$$

Letting $f := \bigotimes_v f_v \in \mathcal{S}(\mathbb{A}_K)$ be defined as the tensor product of all the characteristic functions of $\mathcal{O}_v$ at each place, we have seen from the proof of Proposition 3.1.15 that

$$q^{\dim(L(D))} = \sum_{\gamma \in K} f(\gamma x(D))$$

for any divisor D . By the adelic Riemann-Roch theorem (Theorem 3.1.11), we therefore get that

$$q^{\dim(L(D))} = \frac{1}{|x(D)|_{\mathbb{I}_K}} \sum_{\gamma \in K} \widehat{f}(\gamma x(D)^{-1}).$$

Note that

$$|x(D)|_{\mathbb{I}_K} = \prod_v q_v^{-\text{ord}_v(D)} = q^{-\deg(D)},$$

so it only remains to show that

$$\sum_{\gamma \in K} \widehat{f}(\gamma x(D)^{-1}) = q^{1-g+\dim(L(\mathcal{K}-D))}$$

and that g is a non-negative integer. Recalling our computation of the local Fourier transforms at non-Archimedean places in the proof of Proposition 1.2.14, we see that

$$\widehat{f}_v = N(\text{Conductor of } \psi_v)^{1/2} \mathbb{1}_{\text{Conductor of } \psi_v} = q^{\deg(v) \text{ord}_v(\mathcal{K})/2} \mathbb{1}_{\text{Conductor of } \psi_v},$$

from which it follows that

$$\prod_v \widehat{f}_v = q^{-\deg(\mathcal{K})/2} \bigotimes_v \mathbb{1}_{\text{Conductor of } \psi_v} = q^{1-g} \bigotimes_v \mathbb{1}_{\text{Conductor of } \psi_v}.$$

Thus, for any $\gamma \in K^\times$ we have

$$\widehat{f}(\gamma x(D)^{-1}) = \begin{cases} q^{1-g}, & \text{ord}_v(\gamma) \geq \text{ord}_v(D) + \text{ord}_v(\psi_v) \ \forall v \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\widehat{f}(\gamma x(D)^{-1})$ is nonzero if and only if $\text{div}(\gamma) \in L(\mathcal{K} - D)$, whence the desired equality follows. This also shows that g must be equal to $\dim(L(\mathcal{K}))$, and hence must be a non-negative integer. $\square$

§3.2 The Global Functional Equation

DEFINITION 3.2.1. An **idele-class quasi-character** is a continuous homomorphism $C_K \rightarrow \mathbb{C}^\times$. Equivalently, it is a quasi-character of $\mathbb{I}_K$ trivial on $K^\times$. We will frequently abuse notation between these two.

REMARK. Some texts also use the term **Hecke character** or **Großëncharaktere**, though we stick to the more explicitly descriptive term.

EXAMPLE 3.2.2 (Idelic Lifts of Dirichlet Characters). Recall a **Dirichlet character** of modulus m is a homomorphism $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$. We claim this can be lifted to an idele-class character. Recalling our

decomposition of the rational idele group from Example 1.3.7, one sees immediately that

$$\mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^{\times} \cong \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^{\times} \twoheadrightarrow \widehat{\mathbb{Z}}^{\times} \twoheadrightarrow (\mathbb{Z}/m\mathbb{Z})^{\times} \xrightarrow{\chi} \mathbb{C}^{\times}$$

gives us a character of $C_{\mathbb{Q}}$. A perhaps more non-obvious observation is that *every* quasi-character of $C_{\mathbb{Q}}$ arises in this fashion, up to some complex power of the idelic absolute value. To see this, note we have the exact sequence

$$0 \rightarrow C_{\mathbb{Q}}^1 \rightarrow C_{\mathbb{Q}} \rightarrow |\mathbb{I}_{\mathbb{Q}}|_{\mathbb{A}_{\mathbb{Q}}} \rightarrow 0,$$

and recall by Theorem 1.3.10 that $C_{\mathbb{Q}}^1$ is compact. We can decompose any idele-class quasi-character χ into $\mu\eta$, with $\mu : C_{\mathbb{Q}} \rightarrow S^1$ and $\eta : C_{\mathbb{Q}} \rightarrow \mathbb{R}_{>0}$. Since $\widehat{\mathbb{Z}}^{\times}$ is compact and totally disconnected, any complex quasi-character of it must have finite image, by a standard no-small-subgroups argument.

In the case of η which has its image in the positive reals, this means it must be trivial on $\widehat{\mathbb{Z}}^{\times}$ and therefore factors through the idelic norm $|\cdot|_{\mathbb{I}_{\mathbb{Q}}} : \mathbb{I}_{\mathbb{Q}} \twoheadrightarrow \mathbb{R}_{>0}$. Since any continuous endomorphism of $\mathbb{R}_{>0}$ is an exponential, we deduce η must be of the form

$$\eta(x) = \left(|x|_{\mathbb{I}_{\mathbb{Q}}}\right)^a,$$

for some $a \in \mathbb{R}$.

In the case of μ whose values are in S^1 , it has finite image, and so on $\widehat{\mathbb{Z}}^{\times}$ it must be a lifted Dirichlet character $\tilde{\gamma}$. Moreover its value on $\mathbb{R}_{>0}$ must be some imaginary power of the idelic norm, by a similar argument as with η , using the fact that any character of $\mathbb{R}_{>0}$ corresponds to imaginary exponentiation. Thus we deduce

$$\mu(x) = \left(|x|_{\mathbb{I}_{\mathbb{Q}}}\right)^{bi} \tilde{\gamma}(x),$$

for $b \in \mathbb{R}$ and γ a Dirichlet character. Multiplying them together we get that

$$\chi(x) = \left(|x|_{\mathbb{I}_{\mathbb{K}}}\right)^s \tilde{\gamma},$$

for $s = a + bi \in \mathbb{C}$.

Before we make a definition for our global zeta integral, we must fix a multiplicative measure on the ideles. Recall from (2.1-4), before our analysis on the local theory that, for each completion K_v , our Haar measure $d^{\times}x_v$ on the multiplicative group is $dx_v / |x_v|$ scaled by some constant c_v . We will now see the reason we gave ourselves this flexibility. If we simply chose the easy route setting $c_v = 1$ for all v , then we'd fail to have $\text{Vol}(\mathcal{O}_v^{\times}, d^{\times}x_v) = 1$ for almost all v . More explicitly, the induced measure $d^{\times}x = \prod_v d^{\times}x_v$ on the ideles (in the sense of (1.2-5)) is defective in the following sense: the measure of any basic open set $\prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v^{\times}$ would be

$$\left(\prod_{v \in S} \int_{U_v} d^{\times}x_v\right) \cdot \left(\prod_{v \notin S} \int_{\mathcal{O}_v^{\times}} d^{\times}x_v\right) = \left(\prod_{v \in S} \text{Vol}(U_v, d^{\times}x_v)\right) \cdot \left(\prod_{v \notin S} (1 - N(\mathfrak{p}_v)^{-1})\right),$$

and the second term in the product converges to 0. To amend this, we set our scaling constant (at all the finite places, keeping the infinite ones the same) to

$$c_v = N(\mathfrak{p}_v) / (N(\mathfrak{p}_v) - 1). \quad (3.2-1)$$

Since our standard measure dx_v was normalized so that $\text{Vol}(\mathcal{O}_v, dx_v) = N(\mathfrak{D}_v)^{1/2}$, we have:

$$\text{Vol}(\mathcal{O}_v, dx_v / |x_v|) = (N(\mathfrak{p}_v) - 1) / N(\mathfrak{p}_v) \implies \text{Vol}(\mathcal{O}_v, c_v dx_v / |x_v|) = N(\mathfrak{D}_v)^{1/2}$$

We may then define $d^{\times}x$ on $\mathbb{I}_{\mathbb{K}}$ as the restricted product of all the appropriately scaled local measures, which

makes sense since $\mathcal{O}_v$ is given measure 1 at all the (cofinitely many!) finite unramified places v .

DEFINITION 3.2.3. Given $f \in \mathcal{S}(\mathbb{A}_K)$ and χ an idele-class quasi-character, we define the **global zeta integral** to be

$$Z(f, \chi) := \int_{\mathbb{I}_K} f(x) \chi(x) d^\times x$$

The argument in Example 3.2.2 may be easily generalized to characterize the idele-class quasi-characters for any global field K . Firstly note that any quasi-character trivial on $\mathbb{I}_K^1$ factors through the map $|\cdot|_{\mathbb{I}_K} : \mathbb{I}_K \rightarrow \mathbb{R}_{>0}$, and thus is of the form $|\cdot|^s$ (since any quasi-character on $\mathbb{R}_{>0}$ or $q^\mathbb{Z}$ are of the form $t \mapsto t^s$). Moreover, by the imposition of triviality on $K^\times$, any quasi-character on $\mathbb{I}_K^1$ factors through the compact group C_K^1 , and thus by a "no-small subgroups" argument is a character (i.e S^1 -valued.) Thus any idele-class quasi-character is of the form

$$\chi = \mu |\cdot|_{\mathbb{I}_K}^s \quad (3.2-2)$$

for μ a character on $\mathbb{I}_K^1$ and $s \in \mathbb{C}$. As in §2.1, we denote by σ its exponent (which is intrinsic to χ even though $s \in \mathbb{C}$ is not) and $\chi^\vee$ the shifted dual (see Definitions 2.1.3 and 2.1.4).

As we did in the local theory, by declaring two idele-class quasi-characters to be equivalent if their quotient is trivial on $\mathbb{I}_K^1$, we see an equivalence class consists of quasi-characters of the form

$$\chi = \chi_0 |\cdot|^s,$$

for χ_0 some representative of the equivalence class and $s \in \mathbb{C}$ uniquely determined. Thus we may view the zeta integral as having inputs on the Riemann surface $\mathcal{X}(C_K)$ of idele-class quasi-characters, with local complex coordinate s on each connected component. We now are ready for the crux of Tate's thesis.

THEOREM 3.2.4 (The Global Functional Equation). Let K be a global field. Then for any $f \in \mathcal{S}(\mathbb{A}_K)$ and idele-class quasi-character χ with exponent $\sigma = \Re(\chi)$ the following hold:

- (a) $Z(f, \chi)$ is holomorphic in χ and absolutely converges when $\sigma > 1$, uniformly on $\sigma \geq 1 + \epsilon$, for all $\epsilon > 0$.
- (b) $Z(f, \chi)$ extends to a meromorphic function on all of $\mathcal{X}(C_K)$, with simple poles when $\chi = 1$ and $\chi = |\cdot|_{\mathbb{I}_K}$, with corresponding residues given by $-\text{Vol}(C_K^1)f(0)$ and $\text{Vol}(C_K^1)\hat{f}(0)$.
- (c) The functional equation $Z(f, \chi) = Z(\hat{f}, \chi^\vee)$ is satisfied.

REMARK. In item (b) of the above Theorem, we have not explicitly what measure we are taking the volume of C_K^1 with respect to. The answer is what one might expect: first we induce the measure on $\mathbb{I}_K^1$ by the exact sequence

$$1 \rightarrow \mathbb{I}_K^1 \rightarrow \mathbb{I}_K \rightarrow |\mathbb{I}_K|_{\mathbb{I}_K} \rightarrow 1,$$

where $|\mathbb{I}_K|_{\mathbb{I}_K}$ is given the standard measure $|x|^{-1} dx$ if it is $\mathbb{R}_{>0}$ and the counting measure if it is (some finite-index subgroup of) $q^\mathbb{Z}$, and $\mathbb{I}_K$ is given the measure defined above. The reason for this will become clear in the computations involved in the proof of the above theorem. Then, using that measure on $\mathbb{I}_K^1$ and the counting

measure on $K^\times$, and the exact sequence

$$1 \rightarrow K^\times \rightarrow \mathbb{I}_K^1 \rightarrow C_K^1 \rightarrow 1,$$

we get the measure with respect to which we should compute the volume of C_K^1 . Weil, in [Wei74] normalizes the measure to make $\text{Vol}(C_K^1) = 1$; this is essentially akin to the old dilemma of "where to put the factor of 2π ?" that one encounters in classical Fourier analysis. We elect not to do this and to compute its volume later in terms of the measure scaled locally by the factors described in (3.2-1).

Proof of (a) of 3.2.4. The assertions are $\mathbb{C}$ -linear in f , so it suffices to verify them for the case where $f = \otimes_v f_v$, where f_v is the characteristic function of $\mathcal{O}_v$ for almost all v . Similarly we write χ_v for the restriction of χ the v -th component. Then Corollary 3.1.2 allows us decompose our zeta function into a product of the form:

$$Z(f, \chi) = \prod_{v \in \mathcal{P}_K} Z(f_v, \chi_v).$$

It suffices to show absolute convergence (and uniformity on $\sigma \geq 1 + \epsilon$) of the product of $Z(\mathbb{1}_{\mathcal{O}_v}, |\cdot|_{\mathbb{A}_K}^\sigma) = (1 - |\pi_v|_v^\sigma)^{-1}$ over all the finite places v , since the remaining components will only contribute a finite product.

When K is a number field of degree d over $\mathbb{Q}$, we have (when $\sigma > 1$):

$$1 \leq \prod_{v \nmid \infty} \frac{1}{1 - |\pi_v|_v^\sigma} \leq \prod_{p \text{ prime}} \left(\frac{1}{1 - 1/p^\sigma} \right)^d = \left(\sum_{n \geq 1} \frac{1}{n^\sigma} \right)^d,$$

so we set absolutely convergence with uniformity in this case.

When K has positive characteristic p , expressing it as a finite separable extension of $\mathbb{F}_q(t)$, the relevant product to check for convergence is

$$\prod_{n \geq 1} \left(\frac{1}{1 - 1/q^{n\sigma}} \right)^{N_n}$$

where q is the size of the algebraic closure of $\mathbb{F}_p$ in K and N_n is the number of monic irreducibles of degree n in $\mathbb{F}_q[t]$. The reasoning behind this is that for every monic irreducible of $\mathbb{F}_q[t]$ (we may drop all the places of K above the infinite one on $\mathbb{F}_q(t)$ for the purposes of verifying convergence) there are $\leq [K : \mathbb{F}_q(t)]$ finite places of K over that prime. We know there are $\leq q^n$ monic irreducibles of degree n , so it suffices to check convergence of

$$\prod_{n \geq 1} \left(\frac{1}{1 - 1/q^{n\sigma}} \right)^{q^n}.$$

Taking the formal logarithm gives us

$$\log \left(\prod_{n \geq 1} \left(\frac{1}{1 - 1/q^{n\sigma}} \right)^{q^n} \right) = \sum_{n \geq 1} q^n \log \left(\frac{1}{1 - q^{-n\sigma}} \right).$$

A classical Taylor series expansion tells us $\log(1/(1-x)) \leq x$ (for $x \in (0, 1/2)$, at least, which suffices for our purposes), so the above converges (uniformly on $\sigma \geq 1 + \epsilon$) if the following expression does:

$$\sum_{n \geq 1} q^n q^{-n\sigma} = \sum_{n \geq 1} q^{n(1-\sigma)}.$$

This is visibly convergent when $\sigma > 1$, so thus our original expression is also convergent in this domain, as desired.

Holomorphicity of the function $\chi \mapsto Z(f, \chi)$ then follows by the fact that uniform limits of complex analytic functions are analytic. $\square$

To prove the other parts of the theorem, we employ a classic integration trick: splitting up our integral by norm. We thus write our zeta function as the double integral

$$Z(f, \chi) = \begin{cases} \int_{\mathbb{R}_{>0}} \int_{\mathbb{I}_K^1} f(tx) \chi(tx) d^\times x \frac{dt}{t}, & \text{char}(K) = 0 \\ \sum_{t \in q^{\mathbb{Z}}} \int_{\mathbb{I}_K^1} f(tx) \chi(tx) d^\times x, & \text{char}(K) > 0 \end{cases}$$

where by " tx " we are abusing notation to mean multiplying x by some choice of idele with norm t .

WARNING. When $\text{char}(K) > 0$, we have not yet shown that the image of the idelic norm is in fact all of $q^{\mathbb{Z}}$; this will arrive as a consequence of the proof of our main theorem on the global zeta function! However, we know it is some finite index subgroup of this, so for the purposes of the remainder of the proof we may abuse notation a little and take q to be a *generator* of this subgroup.

The path to proving our desired statements will be via proving a functional equation for the inner integrand

$$Z_t(f, \chi) := \int_{\mathbb{I}_K^1} f(tx) \chi(tx) d^\times x = \int_{\mathbb{I}_K^t} f(x) \chi(x) d^\times x, \quad (3.2-3)$$

where by $\mathbb{I}_K^t$ we mean the ideles of norm t (i.e a coset of $\mathbb{I}_K^1$). The key lemma is as follows.

LEMMA 3.2.5. For any idele-class quasi-character χ and adelic Schwartz-Bruhat function f , the t -local zeta function $Z_t(f, \chi)$ defined in (3.2-3) satisfies

$$Z_t(f, \chi) + f(0) \int_{C_K^1} \chi(tx) d^\times x = Z_{t^{-1}}(\hat{f}, \chi^\vee) + \hat{f}(0) \int_{C_K^1} \chi^\vee(xt^{-1}) d^\times x.$$

WARNING (Notational Abuse). We are writing $d^\times x$ when are actually integrating with respect to the natural quotient measure on C_K^1 discussed in the remark following the Theorem statement of 3.2.4.

Proof. Since $\mathbb{I}_K^1/K^\times = C_K^1$ and χ is trivial on $K^\times$, we have that

$$Z_t(f, \chi) = \int_{C_K^1} \left(\sum_{\alpha \in K^\times} f(\alpha tx) \right) \chi(tx) d^\times x.$$

Here we use discreteness of $K^\times$ in $\mathbb{I}_K$ (Corollary 1.3.2) so that a Haar measure on $K^\times$ is given by the counting measure, then using Fubini to separate the uniform summation from the integral. Again, recall " tx " here simply means multiplying x by some norm- t idele. By Lemma 3.1.8, we know any $f \in \mathcal{S}(\mathbb{A}_K)$ is admissible, so the sum within the integral is (absolutely and uniformly!) convergent. Since we are integrating over a compact set, it follows $Z_t(f, \chi)$ is in fact convergent for any choices of f, χ . This naturally leads us to the following identity of the left side of Lemma 3.1.8:

$$Z_t(f, \chi) + f(0) \int_{C_K^1} \chi(tx) d^\times x = \int_{C_K^1} \sum_{\alpha \in K} f(\alpha tx) \chi(tx) d^\times x.$$

The vital step now is to apply Theorem 3.1.11 – the adelic Riemann-Roch – to the summation over K . More precisely it tells us that $\sum_{\alpha \in K} f(\alpha tx) = 1/|tx|_{\mathbb{I}_K} \sum_{\alpha \in K} \hat{f}(\alpha(xt)^{-1})$. Substituting this in, we get

$$\begin{aligned} Z_t(f, \chi) + f(0) \int_{C_K^1} \chi(tx) d^\times x &= \int_{C_K^1} \chi(tx) \frac{\chi(tx)}{|tx|_{\mathbb{I}_K}} \sum_{\alpha \in K} \hat{f}(\alpha(xt)^{-1}) d^\times x \\ &= \int_{C_K^1} |t^{-1}x|_{\mathbb{I}_K} \chi(tx^{-1}) \sum_{\alpha \in K} \hat{f}(\alpha xt^{-1}) d^\times x \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{I}_K^1} \chi^{-1}(xt^{-1}) \left| xt^{-1} \right|_{\mathbb{I}_K} \widehat{f}(xt^{-1}) d^\times x + \widehat{f}(0) \int_{C_K^1} \chi^{-1}(xt^{-1}) \left| xt^{-1} \right|_{\mathbb{I}_K} d^\times x \\
&= Z_{t^{-1}}(\widehat{f}, \chi^\vee) + \widehat{f}(0) \int_{C_K^1} \chi^\vee(xt^{-1}) d^\times x,
\end{aligned}$$

where the second line follows from a change of variables x to x^{-1} . This proves the functional equation. $\square$

Before continuing, we write out the zeta function as an integral over the t -local zeta functions, splitting it into an integral over the ideles of norm greater than 1, and over the ideles of norm less than 1. This will come in useful in the remainder of the proof.

When K is a number field, we have:

$$Z(f, \chi) = \int_0^1 Z_t(f, \chi) \frac{dt}{t} + \int_1^\infty Z_t(f, \chi) \frac{dt}{t}. \quad (3.2-4)$$

When K is a function field with residue field of order q , we have:

$$Z(f, \chi) = \left(\frac{1}{2} Z_1(f, \chi) + \sum_{n \in \mathbb{Z}_{<0}} Z_{q^n}(f, \chi) \right) + \left(\frac{1}{2} Z_1(f, \chi) + \sum_{n \in \mathbb{Z}_{>0}} Z_{q^n}(f, \chi) \right). \quad (3.2-5)$$

We are now ready for the proof of the remainder of Theorem 3.2.4. Though the proofs in the number field and function field case are practically identical modulo notation, for clarity we divide them up.

Proof of parts (b) and (c) of 3.2.4 in characteristic zero. We first examine the integral over the ideles of norm greater than (or equal to) 1, i.e:

$$\int_1^\infty Z_t(f, \chi) \frac{dt}{t} = \int_{x \in \mathbb{I}_K : |x|_{\mathbb{I}_K} \geq 1} f(x) \chi(x) d^\times x. \quad (3.2-6)$$

By the settled part (a) of our theorem, we know that this converges when $\sigma = \Re(\chi) > 1$. However since all the inputs x of this integral have norm ≥ 1 convergence actually improves when our exponent σ shrinks (recall f is a Schwartz-Bruhat function!). Thus we deduce that (3.2-6) is convergent everywhere.

It remains to analyse the first half of (3.2-4). By our " t -local" functional equation from Lemma 3.2.5, we note that

$$\int_0^1 Z_t(f, \chi) \frac{dt}{t} = \int_0^1 Z_{t^{-1}}(\widehat{f}, \chi^\vee) \frac{dt}{t} + \mathcal{R}(f, \chi), \quad (3.2-7)$$

where $\mathcal{R}$ is the "remainder" given by the functional equation of the lemma, i.e

$$\mathcal{R}(f, \chi) := \int_0^1 \left(\widehat{f}(0) \int_{C_K^1} \chi^\vee(xt^{-1}) d^\times x - f(0) \int_{C_K^1} \chi(tx) d^\times x \right) \frac{dt}{t}.$$

We note that by replacing t with t^{-1} , we get that

$$\int_0^1 Z_{t^{-1}}(\widehat{f}, \chi^\vee) \frac{dt}{t} = \int_1^\infty Z_t(\widehat{f}, \chi^\vee) \frac{dt}{t}.$$

By Proposition 3.1.6, $\widehat{f} \in \mathcal{S}(\mathbb{A}_K)$ and $\chi^\vee$ is by construction also an idele-class quasi-character, so the same argument as for (3.2-6) tells us that this is also convergent everywhere. Thus we have to analyse $\mathcal{R}(f, \chi)$.

If χ is nontrivial on the norm-1 ideles (i.e χ is not in the connected component of $1 \in \mathcal{X}(C_K)$), then integrating either χ or $\chi^\vee$ over C_K^1 (a compact group!) yields zero, that is to say, $\mathcal{R}(f, \chi) = 0$ in this case for any f . Combined with our work done so far, we deduce that in this case $Z(f, \chi)$ is in fact an entire function! In other words, on every connected component of $\mathcal{X}(C_K)$ except the one on which the characters are trivial on $\mathbb{I}_K^1$, the the global zeta integral is holomorphic.

Thus in subsequent analysis we assume that χ is trivial on the norm-1 ideles, or "everywhere unramified", in an analogous sense to Definition 2.1.3. That is to say, $\chi = |\cdot|_{\mathbb{I}_K}^s$, for some $s \in \mathbb{C}$. Thus, we get:

$$\int_{C_K^1} \chi(tx) d^\times x = \int_{C_K^1} |xt|_{\mathbb{A}_K}^s d^\times x = \text{Vol}(C_K^1) t^s.$$

Substituting the above into our remainder factor we get:

$$\begin{aligned} \mathcal{R}(f, \chi) &= \int_0^1 \left(\widehat{f}(0) \text{Vol}(C_K^1) (t^{-1})^{1-s} - f(0) \text{Vol}(C_K^1) t^s \right) \frac{dt}{t} \\ &= \text{Vol}(C_K^1) \int_0^1 \left(\widehat{f}(0) t^{s-1} - f(0) t^s \right) \frac{dt}{t}. \end{aligned}$$

Applying the standard power rule of integration, we get that

$$\mathcal{R}(f, \chi) = \text{Vol}(C_K^1) \left(\frac{\widehat{f}(0)}{s-1} - \frac{f(0)}{s} \right). \quad (3.2-8)$$

This is visibly meromorphic on $\mathbb{C}$ with the appropriate residues at the appropriate simple poles as mentioned in part (b) in the number field case. Finally we need to show the functional equation. The two key identities needed for this are

$$\widehat{\widehat{f}}(x) = f(-x) \text{ and } (\chi^\vee)^\vee = \chi,$$

where the second follows by definition and the first follows by self duality of $\mathbb{A}_K$ with respect to the Tamagawa measure (Corollary 1.2.9). Using this observation and combining the equations in (3.2-4) and (3.2-7), we get

$$\begin{aligned} Z(f, \chi) &= \int_1^\infty Z_t(f, \chi) \frac{dt}{t} + \int_1^\infty Z_t(\widehat{f}, \chi^\vee) \frac{dt}{t} + \mathcal{R}(f, \chi) \\ &= \int_1^\infty \int_{\mathbb{I}_K^1} \widehat{\widehat{f}}(-tx) (\chi^\vee)^\vee(tx) d^\times x \frac{dt}{t} + \int_1^\infty Z_t(\widehat{f}, \chi^\vee) \frac{dt}{t} + \mathcal{R}(\widehat{f}, \chi^\vee) \\ &= \int_1^\infty Z_t(\widehat{\widehat{f}}, (\chi^\vee)^\vee) + \int_1^\infty Z_t(\widehat{f}, \chi^\vee) \frac{dt}{t} + \mathcal{R}(\widehat{f}, \chi^\vee) \\ &= Z(\widehat{f}, \chi^\vee), \end{aligned}$$

where the second line follows from our explicit computation in (3.2-8) showing that $\mathcal{R}(f, \chi)$ is invariant under replacing f with $\widehat{f}$ and χ with $\chi^\vee$, and the third line follows from the substitution $x = -x$ in the inner integrand (implicitly using that $\chi(x) = \chi(-x)$, as χ is trivial on $\mathbb{I}_K^1$). This completes the proof of the functional equation, and in the words of Tate's thesis, "the Main Theorem is proved!" $\square$

We finish by proving the same result in the case of K a function field in positive characteristic.

Proof of parts (b) and (c) of 3.2.4 in positive characteristic. The proof is essentially the same, so we omit some repetitive details. By an identical argument to that used for the equation in (3.2-6), we see that

$$\frac{1}{2} Z_1(f, \chi) + \sum_{n \in \mathbb{Z}_{<0}} Z_{q^n}(f, \chi)$$

(i.e the sum over ideles of norm greater than 1, along with half of Z_1) is convergent everywhere. If t_n denotes a representative of the coset of $\mathbb{I}_K^1$ of norm q^n the t_n -local functional equation gives us

$$\frac{1}{2} Z_1(f, \chi) + \sum_{n \in \mathbb{Z}_{>0}} Z_{q^n}(f, \chi) = \frac{1}{2} Z_1(\widehat{f}, \chi^\vee) + \sum_{n \in \mathbb{Z}_{<0}} Z_{q^n}(\widehat{f}, \chi^\vee) + \mathcal{R}(f, \chi),$$

where

$$\begin{aligned} \mathcal{R}(f, \chi) &= \sum_{n=1}^{\infty} \left(\widehat{f}(0) \chi^{\vee}(t_n^{-1}) \int_{C_k^1} \chi^{\vee}(x) d^{\times} x - f(0) \chi(t_n) \int_{C_k^1} \chi(x) d^{\times} x \right) \\ &\quad + \frac{1}{2} \left(\widehat{f}(0) \int_{C_k^1} \chi^{\vee}(x) d^{\times} x - f(0) \int_{C_k^1} \chi(x) d^{\times} x \right). \end{aligned}$$

This follows from our setup of splitting the integral along the norm-one ideles into half, and the observation that $\chi^{\vee}(t_0^{-1}) = \chi(t_0) = 1$ for any $t_0 \in \mathbb{I}_k^1$, since χ is an idele-class quasi-character trivial on $\mathbb{I}_k^1$ (note triviality on the norm-1 ideles may be assumed by an identical argument to that of the number field case; that the remainder term is zero for any χ nontrivial on this subgroup!). Using the standard geometric series identity then gives us

$$\begin{aligned} \mathcal{R}(f, \chi) &= \frac{\text{Vol}(C_k^1)(\widehat{f}(0) - f(0))}{2} + \sum_{n=1}^{\infty} \text{Vol}(C_k^1) \left(\widehat{f}(0) (q^{-n})^{s-1} - f(0) (q^{-n})^s \right) \\ &= \text{Vol}(C_k^1) \left(\frac{\widehat{f}(0)}{1 - q^{1-s}} - \frac{f(0)}{1 - q^s} + \frac{\widehat{f}(0) - f(0)}{2} \right). \end{aligned} \tag{3.2-9}$$

Here we also see we have the desired residues at the relevant (simple!) poles, thus proving part (b) of the theorem in the function field case. Finally, to show the functional equation we again use the fact that $\widehat{\widehat{f}}(x) = f(-x)$ and $(\chi^{\vee})^{\vee} = \chi$, and the same setup as in the number field case to get:

$$\begin{aligned} Z(f, \chi) &= \frac{1}{2} Z_1(f, \chi) + \sum_{n \in \mathbb{Z}_{<0}} Z_{q^n}(f, \chi) + \frac{1}{2} Z(\widehat{f}, \chi^{\vee}) + \sum_{n \in \mathbb{Z}_{<0}} Z_{q^n}(\widehat{f}, \chi^{\vee}) + \mathcal{R}(f, \chi) \\ &= \frac{1}{2} Z_1(\widehat{\widehat{f}}, (\chi^{\vee})^{\vee}) + \sum_{n \in \mathbb{Z}_{<0}} Z_{q^n}(\widehat{\widehat{f}}, (\chi^{\vee})^{\vee}) + \frac{1}{2} Z(\widehat{f}, \chi^{\vee}) + \sum_{n \in \mathbb{Z}_{<0}} Z_{q^n}(\widehat{f}, \chi^{\vee}) + \mathcal{R}(\widehat{f}, \chi^{\vee}) \\ &= Z(\widehat{f}, \chi^{\vee}). \end{aligned}$$

Now we may once yet again proclaim *"the Main Theorem is proved!"*

□

SECTION 4

EXAMPLES AND APPLICATIONS

§4.1 Immediate Corollaries of the Main Theorem

Here, and in the subsequent sections we go through a number of immediate results that follow from the main result of the prior section. Throughout, K denotes a global field.

PROPOSITION 4.1.1. Let S be a finite set of places of K containing the Archimedean ones (i.e possibly empty when $\text{char}(K) > 0$). Then the product

$$Z_S(s) := \prod_{v \notin S} \frac{1}{1 - N(\mathfrak{p}_v)^{-s}}$$

is absolutely convergent for $\Re(s) > 1$, and moreover $(s - 1)Z_S(s)$ tends to a finite positive limit as $s \rightarrow 1$.

Proof. For absolute convergence, it suffices to verify the product

$$\prod_{v < \infty} \frac{1}{1 - |\pi_v|_v^{\Re(s)}}$$

is convergent for all $\Re(s) > 1$. But this is precisely what we showed in order to prove part (a) of Theorem 3.2.4! For the second assertion, pick some $f \in \mathcal{S}(\mathbb{A}_K)$ so that $f_v = \mathbb{1}_{\mathcal{O}_v}$ for all $v \notin S$ and f_v is the standard function (as tabulated in the fourth column of Table 2.2.A) for all $v \in S$. Then, on the domain $\Re(s) > 1$, we have the decomposition

$$Z(f, |\cdot|_{\mathbb{A}_K}^s) = \prod_{v \in \mathcal{P}_K} Z(f_v, |\cdot|_v^s) = Z_S(s) \prod_{v \in S} Z(f_v, |\cdot|_v^s).$$

The local theory (specifically part (a) of Theorem 2.1.6) tells us that the local zeta functions are in fact absolutely convergent on $\Re(s) > 0$. Moreover, part (b) of Theorem 3.2.4 tells us that the global zeta integral on the left side of the above equality has a simple pole at $s = 1$ with residue $\text{Vol}(C_K^1) \hat{f}(0)$. Thus, we get

$$\lim_{s \rightarrow 1} (s - 1)Z_S(s) = \lim_{s \rightarrow 1} (s - 1) \frac{Z(f, |\cdot|_{\mathbb{A}_K}^s)}{\prod_{v \in S} Z(f_v, |\cdot|_v^s)} = \frac{\text{Vol}(C_K^1) \hat{f}(0)}{\prod_{v \in S} Z(f_v, |\cdot|_v^s)}$$

Examining Table 2.2.A, one sees that $\hat{f}_v(0) > 0$ for all standard f_v , and so $\hat{f}(0) > 0$, and moreover the second column of Table 2.2.B assures us that the local zeta functions with respect to the standard functions are all positive and nonvanishing functions for all $\Re(s) > 0$, and in particular for $s = 1$. Thus we get our desired finite positive limit. $\square$

EXAMPLE 4.1.2 (Hasse-Weil Zeta function). In positive characteristic we have seen the places of K correspond to closed points of the smooth proper curve X geometrically connected over $\mathbb{F}_q$ with function field K . Then the product

$$Z_X(s) := \prod_{v \in X} \frac{1}{1 - |\pi_v|_v^s}$$

is absolutely convergent on the right half plane $\Re(s) > 1$ by Proposition 4.1.1. The above is actually equal to

$$Z_X(s) = \exp \left(\sum_{d=1}^{\infty} \frac{|X(\mathbb{F}_{q^d})|}{d} q^{-ds} \right),$$

where $|X(\mathbb{F}_{q^d})|$ denotes the number of $\mathbb{F}_{q^d}$ -rational points on X . We don't prove this equality here, but this particular generating series, the **Hasse-Weil zeta function**, is of utmost importance, as conjectures about it (first outlined in [Wei49]) and subsequent efforts towards an eventual proof led to the formation of much of modern algebraic geometry. We outline these results here for a general smooth projective variety over $\mathbb{F}_q$.

1. Rationality: For $t := q^{-s}$,

$$\zeta_X(t) = \frac{P_1(t)P_3(t) \cdots P_{2n-1}(t)}{P_0(t)P_2(t) \cdots P_{2n}(t)},$$

where each $P_i(t) \in 1 + t\mathbb{Z}[t]$ for all i .

2. Functional equation:

$$\zeta_X(q^{-n}t^{-1}) = \pm q^{n\chi(X)/2} t^{\chi(X)} \zeta_X(t)$$

where $\chi(X)$ is the Euler characteristic of X .

3. "Riemann hypothesis": For each reciprocal root $\alpha_{i,j}$ of $P_i(t)$, we have $|\alpha_{i,j}| = q^{i/2}$.

4. Betti numbers: If X arises as a "good reduction mod p " of a complex variety, then $\deg P_i = \dim H_i(X)$.

We purposefully occlude some details for brevity and clarity, but we will later see (in §4.3) that the adelic language can be used to prove the Weil conjectures (modulo the Riemann hypothesis) in the case of X a smooth proper geometrically connected curve.

We can use the above proposition and example to prove something we have hinted at all the way back since Chapter I; that the value group of the idelic norm of K in positive characteristic is in fact generated by q

PROPOSITION 4.1.3. For K a global field of positive characteristic and field of constants q , the image of $\mathbb{I}_K$ under the (continuous!) idelic norm is exactly $q^{\mathbb{Z}}$.

Proof. It is clearly some finite index subgroup; it remains to show that we can generate an idele of norm q . Equivalently, if κ_v denotes the residue field of K_v , we want to show that

$$\gcd(\{[\kappa_v : \mathbb{F}_q] : v \in \mathcal{P}_K\}) = 1.$$

Let this gcd be denoted d . Then each κ_v contains a copy of $\mathbb{F}' := \mathbb{F}_{q^d}$ over $\mathbb{F}_q$. Let X be the geometrically connected smooth projective curve with function field K , and consider

$$X' := X \otimes_{\mathbb{F}_q} \mathbb{F}',$$

which is a finite étale connected cover of X that is geometrically connected over $\mathbb{F}'$ and has d points $v_1, \dots, v_d$ over each point $v \in X$. This follows from the fact that

$$\kappa_v \otimes_{\mathbb{F}_q} \mathbb{F}' = \prod_{\sigma: \mathbb{F}' \hookrightarrow \kappa_v} \kappa_v,$$

where the product is taken over all $\mathbb{F}_q$ -linear embeddings of $\mathbb{F}'$ into κ_v , of which there are exactly $[\mathbb{F}' : \mathbb{F}_q] = d$. Thus we have, in the notation of Example 4.1.2,

$$Z_{X'}(s) = (Z_X(s))^d.$$

If $f = \bigotimes_{v \in \mathcal{P}_K} \mathbb{1}_{\mathcal{O}_v}$ and $\chi = |\cdot|_{\mathbb{A}_K}^s$, using the equality discussed in Example 4.1.2 gives us:

$$Z(f, \chi) = Z_X(s).$$

If K' denotes the function field of X' (also a global field), then putting all our observations together tells us that any pole of the zeta function corresponding to K' has order d times the zeta function corresponding to K . However, Theorem 3.2.4 tells us the only poles of the (analytically continued) zeta function of a global field are simple, so we must have $d = 1$! $\square$

REMARK. Interestingly, the above result (at least the way we proved it) can also be used to show the following fact: for K a global field of positive characteristic and $S \subseteq \mathcal{P}_K$ any finite set of places, there exists a divisor $D = \sum n_v \cdot v$ of degree 1 such that $n_v = 0$ for all $v \in S$. We refer the reader to §5 of Chapter VII in [Wei74] for a proof of this.

§4.2 Hecke L -functions and Global Epsilon Factors

The general theory of L -functions on a global field K is of great interest in number theory. One of the primary merits in Tate's approach is the insight into gamma factors and global root numbers of generalized ideal class characters that becomes available via the adelic language. To explore this we will first need to "globalize" the ϵ -factors discussed in Section 2.

In the following propositions, as we did in previous sections, we consider the global ϵ -factor and L -function as having inputs on the Riemann surface $\mathcal{X}(C_K)$ of idele-class quasi-characters.

DEFINITION 4.2.1. Let $(\psi = \prod \psi_v, dx = \prod dx_v)$ be a self-dual pair on $\mathbb{A}_K$. Then we define the **global epsilon factor** to be

$$\epsilon(\chi) := \prod_{v \in \mathcal{P}_K} \epsilon(\chi_v, \psi_v, dx_v).$$

PROPOSITION 4.2.2. The global ϵ -factor in Definition 4.2.1 is a nonvanishing and holomorphic function on $\mathcal{X}(C_K)$, independent of choice of self-dual pair (ψ, dx) .

Proof. We first discuss independence of choice of such a pair of character and measure. Pick a nontrivial adelic character trivial on K and Haar measure

$$\psi = \prod_{v \in \mathcal{P}_K} \psi_v, \quad dx = \prod_{v \in \mathcal{P}_K} dx_v,$$

where we recall from §1.2 that ψ_v is trivial on $\mathcal{O}_v$ and $d^\times x_v$ assigns $\mathcal{O}_v$ measure 1 for almost all places v .

Recall Proposition 1.2.10 tells us $\widehat{\mathbb{A}_K/K} \cong K$, so any other additive character has the form $\psi_b = \psi(bx)$, for some $b \in K^\times$. At each local component, viewing b as an element of $K_v^\times$ under the canonical inclusion, we have seen

that the self-dual local Haar measure relative to $\psi_{v,b}$ is given by $|b|_v^{1/2} dx_v$. By Proposition 2.2.1 (using Corollary 2.2.3 to apply it to the ϵ -factors), we get:

$$\epsilon(\chi_v, \psi_{v,b}, |b|_v^{1/2} dx_v) = \chi_v(b) |b|_v^{-1/2} \epsilon(\chi_v, \psi_v, dx_v).$$

Applying this over the product (which also verifies that the restricted product of these modified local measures is a valid measure globally) gives us:

$$\begin{aligned} \prod_{v \in \mathcal{P}_K} \epsilon(\chi_v, \psi_{v,b}, |b|_v^{1/2} dx_v) &= \prod_{v \in \mathcal{P}_K} \chi_v(b) |b|_v^{-1/2} \epsilon(\chi_v, \psi_v, dx_v) \\ &= \chi(b) |b|_{\mathbb{A}_K}^{-1/2} \prod_{v \in \mathcal{P}_K} \epsilon(\chi_v, \psi_v, dx_v) \\ &= \prod_{v \in \mathcal{P}_K} \epsilon(\chi_v, \psi_v, dx_v), \end{aligned}$$

where the final equality follows from the fact that χ is trivial on $K^\times$ and the product formula (1.3.6) telling us $|b|_{\mathbb{A}_K} = 1$. Thus, $\epsilon(\chi)$ is independent of choice of self-dual pair.

For nonvanishing and holomorphicity, examining our record of all local ϵ -factors in Table 2.2.B, (combined with Proposition 2.2.1) we see that they are non-vanishing and holomorphic for any choice of local χ_v, ψ_v, dx_v , so it remains only to check convergence.

Any $\chi \in \mathcal{X}(C_K)$ is unramified at almost all places, so examining the fourth row of Table 2.2.B tells us that at cofinitely many v the local epsilon factor is of the form

$$q^{|m_v|s} \text{Vol}(\mathfrak{p}_v^{m_v}, dx_v),$$

where $\mathfrak{p}_v^{m_v}$ is the conductor of ψ_v . However, we also recall that any adelic character must have trivial conductor (i.e $m_v = 0$) at almost all places, and that any adelic measure must assign $\mathcal{O}_v$ volume 1 at almost all places. Thus, for any χ , the product $\epsilon(\chi)$ reduces (locally, on $\mathcal{X}(C_K)$) to a finite product of non-vanishing holomorphic functions, thereby proving our claim. $\square$

We also introduce here the notion of a *global* root number.

DEFINITION 4.2.3. Given an idele-class quasi-character χ of exponent 1/2, its **global root number** is the complex number given by

$$W(\chi) := \prod_{v \in \mathcal{P}_K} W(\chi_v),$$

where each $W(\chi_v)$ is the local root number as in Definition 2.2.4.

SANITY CHECK. The definition above makes sense, since the exponent of each local χ_v will also be 1/2, as was required in Definition 2.2.4, and at almost all places we will have $W(\chi_v) = 1$, which we leave the reader to verify. Again, we also omit choice of adelic character or self-dual measure, since the root number is independent of them by construction. Moreover, since all the local root numbers have norm 1, so does the global root number.

This is the factor that comes up in the functional equation of the classical Dirichlet L -function, which we will see later in Example 4.2.7. Tate's adelic formulation gives a meaningful decomposition of this factor.

With the ϵ -factors and root numbers behind us, we may proceed to globalize the L -function. Take some idele-class quasi-character $\chi \in \widehat{C_K}$, and – viewing it as a character on $\mathbb{A}_K$ – recall we can consider the local restrictions

χ_v for each place v . We can then define the **global L -function** by taking the product of all the local factors, i.e.

$$L(\chi) := \prod_{v \in \mathcal{P}_K} L(\chi_v). \quad (4.2-1)$$

It will be useful to record here the local L -factors and compare their relation to local zeta functions for some choice of local standard function f_v . We recall here the "specialized" Gamma functions:

$$\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s} \Gamma(s), \quad \Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2),$$

where Γ is the classical analytic extension of the factorial function.

	Multiplicative χ	Local $L(\chi)$	Local zeta function $Z(f, \chi)$	Quotient $Z(f, \chi)/L(\chi)$
$F = \mathbb{R}$	$\chi = \cdot ^s$	$\Gamma_{\mathbb{R}}(s)$	$\Gamma_{\mathbb{R}}(s)$	1
	$\chi = \text{sgn} \cdot ^s$	$\Gamma_{\mathbb{R}}(s+1)$	$\Gamma_{\mathbb{R}}(s+1)$	1
$F = \mathbb{C}$	$\chi_{s,n}$	$\Gamma_{\mathbb{C}}\left(s + \frac{ n }{2}\right)$	$\Gamma_{\mathbb{C}}\left(s + \frac{ n }{2}\right)$	1
F non-Arch	$\chi_{s,n}, n = 0$ (unramified)	$(1 - \pi_F ^{-s})^{-1}$	$\frac{q^{-ms}}{1-q^{-s}} \text{Vol}(\mathcal{O}_F^\times, d^\times x)$	$q^{-ms} \text{Vol}(\mathcal{O}_F^\times, d^\times x)$
	$\chi_{s,n}, n > 0$ (ramified)	1	$q^{(n-m)s} \text{Vol}(\mathcal{O}_F, dx) \text{Vol}(1 + \mathfrak{p}^n, d^\times x) \overline{g(\omega, \psi_{\pi^{m-n}})}^{-1}$	$Z(f, \chi)$.

Table 4.2.A: On the strip $0 < \Re(\chi) < 1$ of each connected component of $\mathcal{X}(F^\times)$ the zeta integral is computed with respect to the corresponding additive character, measure and standard function outlined in Table 2.2.A. The zeta integrals themselves were tabulated in Table 2.2.B. At non-Archimedean F , the q denotes the size of its residue field, m denotes the conductor of the (variable) additive character ψ , and ω denotes the unramified part of χ (which depends on choice of homomorphic retraction of $F^\times$ to $\mathcal{O}_F^\times$).

Before stating the main theorem on these L -functions we recall the **different ideal** $\mathfrak{d}_K$ of a global field K . In the number field case, this is a proper ideal of $\mathcal{O}_K$ whose norm recovers $|\text{disc}(\mathcal{O}_K)|$. In the function field case, the different is better thought of as a *divisor* (since there is no canonical ring of integers) dependent on our choice of standard adelic character ψ , which itself was dependent on some choice of nonzero global meromorphic 1-form $\omega \in \Omega_{K/\mathbb{F}_q}^1$. The degree of this divisor d_v at each place v is the order of ω at v . Moreover, in positive characteristic by the **different** $\mathfrak{D}_v$ of K_v we mean the inverse fractional ideal of conductor of ψ_v , which is precisely the ideal $\mathfrak{p}_v^{-d_v}$.

THEOREM 4.2.4. For an idele-class quasi-character χ the global L -function $L(\chi)$ converges (absolutely and uniformly on $\Re(\chi) \geq 1 + \epsilon$ for every $\epsilon > 0$) when $\Re(\chi) > 1$, and has a meromorphic continuation to all of $\mathcal{X}(C_K)$, holomorphic outside of $\chi = 1$ and $\chi = |\cdot|_{\mathbb{I}_K}$, where it has respective residues $-(2\pi)^{-r_2} \text{Vol}(C_K^1)$ and $(2\pi)^{-r_2} \text{Vol}(C_K^1)$.

Proof. All following computations in this proof will be done using self-dual pair (ψ, dx) of the standard character and Tamagawa measure with respect to which we defined the Fourier transform, all the way back in (3.1-2). We will see that this is no loss of generality.

For convergence, examining Table 4.2.A shows us that $L(\chi_v)$ is a well-defined (and nonvanishing!) on the domain $\Re(\chi) > 1$ at all the Archimedean places, and all the ramified non-Archimedean ones (of which there are

only finitely many). Thus it suffices to verify that the product

$$\prod_{v < \infty} \frac{1}{1 - \chi_v(\pi_v)^{\Re(\chi)}}$$

is convergent on the domain $\Re(\chi) > 1$; this is precisely the statement of Proposition 4.1.1 with $S = S_\infty$.

To show meromorphic continuation, it will suffice to produce a function $f \in \mathcal{S}(\mathbb{A}_K)$ and G_f on $\mathcal{X}(C_K)$ non-vanishing and holomorphic on $\Re(\chi) > 1$ so that

$$L(\chi) = Z(f, \chi) / G_f(\chi)$$

for $\Re(\chi) > 1$, since Theorem (a) of 3.2.4 assures us $Z(f, \chi)$ is convergent on this domain and part (b) tells us it admits meromorphic continuation to all of $\mathcal{X}(C_K)$.

First we construct f as the "adelic tensor product" of all the standard functions f_v described in Table 2.2.A. In particular we have

- $f_v = e^{-\pi x^2}$, for v real.
- $f_v = 1/(2\pi)e^{-2\pi z\bar{z}}$, for v complex.
- $f_v = \mathbb{1}_{\mathfrak{p}_v^{m_v}}$, for v non-Archimedean (recall m_v is the conductor of ψ_v).

This is valid since each $f_v \in \mathcal{S}(K_v)$, and moreover using the fact that the conductors of ψ_v, χ_v are trivial at almost all places, we get that $f_v = \mathbb{1}_{\mathcal{O}_v}$ for cofinitely many v . Hence $\bigotimes_v f_v \in \mathcal{S}(\mathbb{A}_K)$.

Furthermore, we can explicitly construct G_f by defining its local components $G_{f,v}$, and the fourth-column of Table 4.2.A tells us exactly what they should be. In particular the local components of G_f are given by:

- $G_{f,v} = 1$, for v Archimedean
- $G_{f,v} = q^{-ms} \text{Vol}(\mathcal{O}_F^\times d^\times x)$, for non-Archimedean v where χ_v is unramified.
- $G_{f,v} = Z(f_v, \chi_v)$, for non-Archimedean v where χ_v is ramified.

By construction then we get $L(\chi_v)G_{f,v} = Z(f_v, \chi_v)$ and moreover each $G_{f,v}$ is visibly nonvanishing and holomorphic on all of $\mathcal{X}(K_v^\times)$. We also recall that multiplicative χ is unramified at cofinitely many places. Of those, (at the unramified places of K) cofinitely many have trivial conductor for the local additive character ψ_v . In other words, $m_v = 0$ at almost all v , and recall our local scaling constant for the measures on $K_v^\times$ in (3.2-1) assigns each $\mathcal{O}_v^\times$ measure $N(\mathfrak{D}_v)^{-1/2} = 1$ at all these places as well. It follows the function

$$G_f(\chi) := \prod_{v \in \mathcal{P}_K} G_{f,v}(\chi_v)$$

is well-defined, nonvanishing and holomorphic on all of $\mathcal{X}(C_K)$, and by construction satisfies

$$L(\chi)G_f(\chi) = Z(f, \chi)$$

on this domain. Thus, $L(\chi)$ admits meromorphic continuation to all of $\mathcal{X}(C_K)$.

It remains to verify the poles and residues. Since we have $L(\chi) = Z(f, \chi)/G_f(\chi)$, with G_f non-vanishing and holomorphic, the poles of $L(\chi)$ are exactly those of $Z(f, \chi)$, and the residue at any χ_0 is that of $Z(f, \chi_0)$ multiplied by $1/G_f(\chi_0)$. Theorem 3.2.4 tells us $Z(f, \chi)$ has simple poles at $\chi = 1$ and $\chi = |\cdot|_{\mathbb{A}_K}$, with respective residues $-\text{Vol}(C_K^1)f(0)$ and $\text{Vol}(C_K^1)\widehat{f}(0)$. Noting that that both these poles are unramified characters and examining

Table 2.2.A, we see that $f_v(0) = 1$ for all non-complex places, and thus $f(0) = (2\pi)^{-r_2}$. Moreover at $\chi = 1$ we have $1/G_f(\chi) = 1$. Thus, we get

$$\text{Res}_{\chi=1} L(\chi) = -(2\pi)^{-r_2} \text{Vol}(C_K^1),$$

To compute the residue at $\chi = |\cdot|_{\mathbb{I}_K}$, we will need to compute the Fourier transform of the standard function f . We recall from construction the the standard ψ_v has conductor exactly $\mathfrak{D}_v^{-1}$ (for function fields this is true by definition) for all non-Archimedean v . Mimicking our proof of Proposition 1.2.14, we have

$$\begin{aligned} \widehat{f}_v(y) &= \int_{\mathfrak{D}_v^{-1}} \overline{\psi_v(xy)} \, dx_v \\ &= \begin{cases} \text{Vol}(\mathfrak{D}_v^{-1}, dx_v), & y \in \mathfrak{D}_v, \\ 0, & y \notin \mathfrak{D}_v. \end{cases} \end{aligned}$$

Recall from construction of the Tamagawa measure (see the local components after Definition 1.2.13) that $\text{Vol}(\mathcal{O}_v, dx_v) = N(\mathfrak{D}_v)^{-1/2}$, whence it follows taking the volume of the codifferent gives us $N(\mathfrak{D}_v^{1/2})$. With this and the classical results in the Archimedean cases, we have:

- $\widehat{f}_v = f_v$, for v real.
- $\widehat{f}_v = f_v$, for v complex.
- $\widehat{f}_v = (N\mathfrak{D}_v)^{1/2} \mathbb{1}_{\mathfrak{D}_v}$, for v non-Archimedean.

Multiplying them all out together gives us

$$\widehat{f}(0) = \prod_{v \in \mathcal{P}_K} \widehat{f}_v(0) = (2\pi)^{-r_2} \prod_{v < \infty \text{ ramified}} (N\mathfrak{D}_v)^{1/2} = (2\pi)^{-r_2} |\mathfrak{d}_K|^{1/2}. \quad (4.2-2)$$

At $\chi = |\cdot|_{\mathbb{I}_K}$, the local factors are all trivial except at the non-Archimedean places, where

$$G_{f,v}(|\cdot|_{\mathbb{I}_K}) = q_v^{-m_v} \text{Vol}(\mathcal{O}_v^\times, d^\times x_v).$$

Again we use the fact that at all non-Archimedean places we have the conductor of ψ_v is $\mathfrak{p}_v^{m_v} = \mathfrak{D}_v^{-1}$ and that dx_v assigns $\mathcal{O}_v$ the value $(N\mathfrak{D}_v)^{-1/2} = q_v^{m_v/2}$. Thus,

$$G_{f,v}(|\cdot|_{\mathbb{I}_K}) = q_v^{-m_v} q_v^{m_v/2} = q_v^{-m_v/2} = (N\mathfrak{D}_v)^{1/2}$$

Multiplying out all the local components (again using the fact that $m_v = 0$ if and only if v is unramified) therefore gives us

$$\frac{1}{G_f(|\cdot|_{\mathbb{I}_K})} = \prod_{v < \infty \text{ ramified}} (N\mathfrak{D}_v)^{-1/2} = |\mathfrak{d}_K|^{-1/2} \quad (4.2-3)$$

Putting (4.2-2) and (4.2-3) together we get

$$\text{Res}_{\chi=|\cdot|_{\mathbb{I}_K}} L(\chi) = (2\pi)^{-r_2} \text{Vol}(C_K^1),$$

completing our proof. □

THEOREM 4.2.5. For any idele-class quasi-character χ , we have

$$L(\chi) = \epsilon(\chi) L(\chi^\vee).$$

Proof. In all proceeding computations, we let (ψ, dx) be the self-dual pair of the restricted product of the characters and measures outlined in Table 2.2.A. Proposition 4.2.2 assures us this is no loss of generality.

Take any $\chi \in \mathcal{X}(C_K)$. Examining the fourth column of Table 4.2.A, we see that, at all the (cofinitely!) many places where χ_v is unramified and ψ_v has conductor $\mathcal{O}_v$, we have

$$L(\chi_v) = Z(f_v, \chi_v) \quad \text{and} \quad L(\chi_v^\vee) = Z(\hat{f}_v, \chi_v^\vee).$$

Let S be the finite set of places where this is *not* the case, and define

$$L_S(\chi) = \prod_{v \notin S} L(\chi_v).$$

When $\Re(\chi) > 1$, we then have the following decomposition:

$$Z(f, \chi) = L_S(\chi) \prod_{v \in S} Z(f_v, \chi_v).$$

Noting that all the local $Z(f_v, \chi_v)$ are nonvanishing holomorphic functions on $\Re(\chi_v) > 0$, since the right side of the above expression admits a meromorphic continuation to all of $\mathcal{X}(C_K)$ (part (b) of 3.2.4), so does $L_S(\chi)$.

Now let us work on the domain $\Re(\chi) < 0$ (so $\Re(\chi^\vee) > 1$). Part (c) of Theorem 3.2.4 tells us we have

$$Z(f, \chi) = Z(\hat{f}, \chi^\vee),$$

with the left side being the meromorphic continuation of the zeta function and the right side decomposing into an absolutely convergent product. Thus, we get:

$$\begin{aligned} L_S(\chi) \prod_{v \in S} Z(f_v, \chi_v) &= \prod_{v \in \mathcal{P}_K} Z(\hat{f}_v, \chi_v^\vee) \\ &= L_S(\chi^\vee) \prod_{v \in S} \epsilon(\chi_v) \frac{L(\chi_v^\vee)}{L(\chi_v)} Z(f_v, \chi_v) \\ \implies L_S(\chi) &= L(\chi^\vee) \prod_{v \in S} \frac{\epsilon(\chi_v)}{L(\chi_v)} \\ \implies L(\chi) &= \epsilon(\chi) L(\chi^\vee), \end{aligned}$$

where the final equality follows from our work in the proof of Proposition 4.2.2 that showed $\epsilon(\chi_v, \psi_v, dx_v) = 1$ at all places v where χ_v is unramified and ψ_v has trivial conductor, i.e all $v \notin S$, thus implying $\epsilon(\chi) = \prod_{v \in S} \epsilon(\chi_v)$. This completes our proof. $\square$

Classically, the L -function is written in the manner given in the following definition.

DEFINITION 4.2.6. For χ an idele-class quasi-character, the **Hecke L -function** is

$$L(s, \chi) := L(\chi | \cdot|^s).$$

In the more classical notation of the Hecke the functional equation proved in 4.2.4 is given by

$$L(s, \chi) = \epsilon(s, \chi) L(1 - s, \chi^{-1}),$$

where naturally we define $\epsilon(s, \chi) := \epsilon(\chi | \cdot|^s)$. It was this result (though not in this exact formulation) that Tate concluded his thesis with, to display how the adelic formulation gave rise to elegant arguments for such classically known results.

EXAMPLE 4.2.7 (Functional equation of the classical Dirichlet L -function). Suppose $K = \mathbb{Q}$, and let χ_0 be a *primitive* Dirichlet character with conductor $N > 1$; we have seen from Example 3.2.2 that we may pick an idelic lift χ of χ_0 so that χ has exponent 0, which would tell us that $\chi^{-1} = \bar{\chi}$. By Theorem 4.2.5, we deduce that

$$L(s, \chi) = \epsilon(s, \chi) L(1-s, \bar{\chi}).$$

Our objective here will be to compute $\epsilon(s, \chi)$. On all the local factors we use the standard characters from Definition 1.2.2 (i.e has trivial conductor at all the finite places) and the self-dual local measure with respect to these characters (i.e Lebesgue on the infinite place and assigning volume 1 to all $\mathbb{Z}_p$). If $N_p := \text{ord}_p(n)$, we make the observation that χ_p has conductor $p^{N_p} \mathbb{Z}_p$ at all the finite places. Thus examining Table 2.2.B, we see that

$$\begin{aligned} \epsilon(\chi_\infty) &= i^\delta \\ \epsilon(\chi_p) &= \begin{cases} 1, & p \nmid N \\ p^{-N_p(s-1)} g(\bar{\omega}_p, \psi_{p^{-N_p}}), & p \mid N \end{cases} \end{aligned}$$

where here $\delta = 0$ if $\chi_0(-1) = 1$ and 1 otherwise. Expanding out the Gauss sum at all the primes p where χ_p is ramified (i.e precisely the primes dividing N) we have:

$$\begin{aligned} g(\bar{\omega}_p, \psi_{p^{-N_p}}) &= \omega_p(-1) g(\omega_p, \psi_{p^{-N_p}}) \\ &= \omega_p(-1) \int_{\mathbb{Z}_p^\times} \omega_p(u) \psi(up^{-N_p}) d^\times u. \end{aligned}$$

Since χ has conductor N_p , it factors through the quotient $\mathbb{Z}_p^\times / U_{N_p} \cong (\mathbb{Z}/p^{N_p}\mathbb{Z})^\times$, where

$$U_{N_p} := 1 + p^{N_p} \mathbb{Z}_p$$

Recalling our standard character on $\mathbb{Q}_p$ defined in (1.2-2), and writing $\bar{u}$ for the image of $u \in \mathbb{Z}_p^\times$ under the quotient map, we see that the above is equal to:

$$\omega_p(-1) \sum_{\bar{u} \in (\mathbb{Z}/p^{N_p}\mathbb{Z})^\times} \omega_p(\bar{u}) e^{2\pi i \bar{u} p^{-N_p}} \left(\int_{U_{N_p}} \psi(up^{-N_p}) d^\times u \right)$$

Note for any $u = 1 + u_0 p^{N_p} \in U_{N_p}$ we have $p^{N_p} u = p^{-N_p} + u_0 \in \mathbb{Z}_p^\times$. Since the conductor of each local additive character is $\mathbb{Z}_p$, we have that $\psi_{p^{-N_p}}(u) = 1$ for all $u \in U_{N_p}$, so we have:

$$\begin{aligned} g(\bar{\omega}_p, \psi_{p^{-N_p}}) &= \omega_p(-1) \sum_{\bar{u} \in (\mathbb{Z}/p^{N_p}\mathbb{Z})^\times} \omega_p(\bar{u}) e^{2\pi i \bar{u} p^{-N_p}} \text{Vol}(U_{N_p}, d^\times u) \\ &= \omega_p(-1) p^{-N_p} \sum_{\bar{u} \in (\mathbb{Z}/p^{N_p}\mathbb{Z})^\times} \chi_0(\bar{u}) e^{-2\pi i \bar{u} p^{-N_p}} \end{aligned}$$

If we define $\tau_p(\chi_0) := \sum_{\bar{u} \in (\mathbb{Z}/p^{N_p}\mathbb{Z})^\times} \chi_0(\bar{u}) e^{-2\pi i \bar{u} p^{-N_p}}$, then by primitivity of χ_0 we get:

$$\prod_{p|N} \tau_p(\chi_0) = \sum_{k=1}^N \chi_0(k) e^{-2\pi i k/N} =: \tau(\chi_0),$$

where here $\tau(\chi_0)$ denotes the classical **Gauss sum** of a Dirichlet character. Using this fact and multiplying

all the local ϵ -factors together, we get:

$$\begin{aligned}\epsilon(s, \chi) &= i^\delta \prod_{p|N} p^{-N_p(s-1)} \omega_p(-1) p^{-N_p} \tau_p(\chi_0) \\ &= i^\delta \chi_0(-1) N^{-s} \tau(\chi_0).\end{aligned}$$

Using the observation that $\chi(-1)i^\delta = (-i)^\delta$ and substituting the above expression into the functional equation gives us:

$$L(s, \chi) = N^{-s} (-i)^\delta \tau(\chi_0) L(1-s, \bar{\chi}).$$

Note that the factors of $(-i)^\delta$ and $\tau(\chi_0)$ are independent of s , so in particular if $s = 1/2$ we would have $\epsilon(1/2, \chi) = W(\chi)$, as in Definition 4.2.3. Thus, we can write

$$L(s, \chi) = N^{-s+1/2} W(\chi) L(1-s, \bar{\chi}).$$

Multiplying both sides by $N^{s/2}$ and setting $\Lambda(s, \chi) := N^{s/2} L(s, \chi)$, we get:

$$\Lambda(s, \chi) = W(\chi) \Lambda(1-s, \bar{\chi}),$$

which is precisely the functional equation of the completed Dirichlet L -function!

SANITY CHECK. In the above example, we should check that $\Lambda(s, \chi)$ and $W(\chi)$ deserve their notation, in the sense that our adelic definitions align with the classical ones. For the root number, our adelic language gave us

$$W(\chi) = N^{-1/2} (-i)^\delta \tau(\chi_0),$$

but this is precisely the classical definition of the root number! At all the primes coprime to N the local character is unramified, we literally have $L(s, \chi_p) = (1 - \chi(p)p^{-s})^{-1}$, and at all the primes dividing N the local L -factor is trivial, so we get the classical (non-completed) Euler product

$$L(s, \chi^f) = \prod_p L(s, \chi_p),$$

where here $L(s, \chi^f)$ denotes the classical Dirichlet L -function. At the (real) infinite place, we note that

$$L(s, \chi_\infty) = \pi^{(s+\delta)/2} \Gamma\left(\frac{s+\delta}{2}\right),$$

And so we see

$$\Lambda(s, \chi) := N^{s/2} L(s, \chi) = N^{s/2} \pi^{(s+\delta)/2} \Gamma\left(\frac{s+\delta}{2}\right) L(s, \chi^f),$$

which we see agrees exactly with the classical completion of the Dirichlet L -function.

§4.3 Dedekind Zeta Functions

DEFINITION 4.3.1. For K a global field with trivial idele-class quasi-character $\chi = 1$, the product of all the non-Archimedean local L -factors is called the **Dedekind zeta function**, and denoted

$$\zeta_K(s) = \prod_{v \text{ non-Arch.}} \frac{1}{1 - |\pi_v|_v^{-s}}.$$

In other words, when K is a number field the product is taken over all the finite places, and when K is a global function field it is taken over all places. We note that unique prime factorization of ideals in a Dedekind domains tells us, via a computation similar to classical Euler expansion,

$$\zeta_K(s) = \prod_{v \text{ non-Arch.}} \frac{1}{1 - (N(\mathfrak{p}_v))^{-s}} = \prod_{v \text{ finite}} \left(\sum_{n \in \mathbb{N}} \frac{1}{N(\mathfrak{p}_v)^{ns}} \right) = \sum_{\mathfrak{a} \in \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}. \quad (4.3-1)$$

EXAMPLE 4.3.2. When $K = \mathbb{Q}$ then (4.3-1) in the above observation tells us

$$\zeta_{\mathbb{Q}}(s) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n|} = \sum_{n \geq 1} \frac{1}{n^s},$$

i.e. we recover the classical Riemann zeta function.

Our goal will be to give explicit characterizations of the Dedekind zeta function for both the number field and function field case. We begin with the case of the number fields. Before we do so, however, we need to recall a few objects from algebraic number theory.

Suppose K has r_1 real embeddings and r_2 complex embeddings (up to complex conjugation), each corresponding to an infinite place $v \in S_{\infty} := \mathcal{P}_{K,\infty}$. By Dirichlet's unit theorem (Example 1.4.4, which itself is a consequence of the generalized S -unit theorem), the unit group $\mathcal{O}_K^{\times}$ has rank $r - 1$, where $r := r_1 + r_2$.

DEFINITION 4.3.3. For K a number field, let $u_1, \dots, u_{r-1}$ be a set of generators for $\mathcal{O}_K^{\times}/\mu(K)$, and for each $v \in S_{\infty}$, let N_v be 1 if v is real and 2 if it is complex. Consider the $(r-1) \times r$ matrix

$$M := [N_v \log(|u_i|_v)]_{v \in S_{\infty}, i \in [r-1]}$$

and let M' be the $(r-1) \times (r-1)$ matrix obtained by deleting any one column of M . Then the **regulator** of K is defined as $|\det(M)|$.

Essentially, the regulator is a measure of the "relative size" of the group of units of a number field. The larger the regulator, the "fewer" units there are. That the regulator is well-defined is a consequence of the fact that the sum of any row is zero, which itself is a consequence of Lemma 4.3.5 which is used in the proof of Theorem 4.3.4. Geometrically, what is happening is we are taking embedding the group of units as a lattice in a hyperplane in $\mathbb{R}^{r_1+r_2}$, and then taking its "covolume", i.e. the volume of the hyperplane quotiented out by the lattice in standard Euclidean Haar measure.

THEOREM 4.3.4. Let K be a number field with r_1 real places, r_2 complex places, root of unity group $\mu(K)$, regulator R_K , class number h_K , and discriminant $\mathfrak{d}_K$. Then the function $\zeta_K(s)$ is holomorphic except for simple poles at $s = 0$ and 1 with respective residues $-(2\pi)^{-r_2} c_K$ and c_K , where

$$c_K := \frac{2^{r_1} (2\pi)^{r_2} h_K R_K}{|\mu(K)| |\mathfrak{d}_K|^{1/2}}.$$

For convenience we again recall here the specialized Gamma factors:

$$\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s} \Gamma(s), \quad \Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2).$$

Note that

$$L(|\cdot|_{\mathbb{I}_K}^s) = L(s, 1) = (\Gamma_{\mathbb{R}}(s))^{r_1} (\Gamma_{\mathbb{C}}(s))^{r_2} \zeta_K(s).$$

By Theorem 4.2.4, this has residue $-(2\pi)^{r_2} \text{Vol}(C_K^1)$ at $s = 0$ and $(2\pi)^{-r_2} \text{Vol}(C_K^1)$ at $s = 1$. We see that

$$\Gamma_{\mathbb{C}}(0) = \Gamma_{\mathbb{R}}(0) = 1,$$

so these factors don't affect the residue at $s = 0$, i.e the residue of $L(s, 1)$ at 0 must be equal to that of $\zeta_K(s)$. At 1, we see

$$\Gamma_{\mathbb{C}}(1) = (2\pi)^{-1}, \quad \Gamma_{\mathbb{R}}(1) = 1.$$

Thus the complex gamma factors contribute a residue of $(2\pi)^{-r_2}$ to the global L -function, so it follows $\zeta_K(s)$ must have residue $\text{Vol}(C_K^1)$ at $s = 1$. Hence, all we have left to do is show that $\text{Vol}(C_K^1)$ (with respect to the measure induced appropriately scaled $d^\times x$ on $\mathbb{I}_K$ and the counting measure on K) is equal to c_K .

Let S denote any finite set of places of K containing the infinite ones, and recall from Definition 1.4.1 the subgroup of S -ideles $\mathbb{I}_{K,S}$. We define the S -idelic logarithm map by

$$\begin{aligned} \mathcal{L}_S : \mathbb{I}_{K,S}^1 &\rightarrow \mathbb{R}^{|S|} \\ (x_v)_v &\mapsto (\log |x_v|_v)_{v \in S}. \end{aligned} \tag{4.3-2}$$

LEMMA 4.3.5. When $S = S_\infty$, define the hyperplane

$$H_S := \left\{ (t_v)_{v \in S} : \sum_{v \text{ real}} t_v + 2 \sum_{v \text{ complex}} t_v = 0 \right\}.$$

Then $\text{Im}(\mathcal{L}_S) = H_S$ and $\ker(\mathcal{L}_S) = \mathbb{I}_{K,\emptyset}$.

Proof. By definition, $\prod_{v \in S} |x_v|_v = 1$, so thus $\sum_{v \in S} \log(|x_v|_v) = 0$, which is precisely the equation defining the hyperplane H_S (recall our normalized absolute value on $\mathbb{C}$ defined in Definition 1.3.4 is the square of the standard one). Moreover given any $(t_v)_v \in H$ we can construct an element of $\mathbb{I}_{K,S}^1$ by setting x_v of absolute value e^{t_v} for $v \in S$ and picking some $x \in \mathcal{O}_v^\times$ for all the $v \notin S$. This proves the image of the logarithm is exactly H_S . For the kernel, note $(x_v)_v \in \ker(\mathcal{L}_S)$ if and only if $|x_v|_v = 1$ (which is true if and only if $x_v \in \mathcal{O}_v^\times$) for all $v \in S$. Thus, $(x_v)_v \in \prod_v \mathcal{O}_v^\times = \mathbb{I}_{K,\emptyset}^1 = \mathbb{I}_{K,\emptyset}$. $\square$

REMARK. For the reasons discussed following Definition 4.3.3, Ramakrishnan [RV98] called the restriction of $\mathcal{L}_{S_\infty}$ to $\mathcal{O}_K^\times$ the *regulator map*.

Proof of Theorem 4.3.4. Let $S = S_\infty$ be the (nonempty, since we are in characteristic zero) set of Archimedean places of K . Recall the S -class group $C_{K,S}$, which, from the observation following Definition 1.4.7, using the fact that S is nonempty we may write

$$C_{K,S} = \mathbb{I}_K^1 / (K^\times \cdot \mathbb{I}_{K,S}^1).$$

We make the observation that the kernel of the projection $C_K^1 \twoheadrightarrow C_{K,S}$ is precisely $\mathbb{I}_{K,S}^1 / (K^* \cap \mathbb{I}_{K,S}^1)$. By Proposition, 1.4.8, since S is nonempty we deduce $C_{K,S}$ is a finite group, hence we have that

$$\text{Vol}(C_K^1) = |C_{K,S}| \cdot \text{Vol}(\mathbb{I}_{K,S}^1 / (K^* \cap \mathbb{I}_{K,S}^1)). \tag{4.3-3}$$

Since $S = S_\infty$, we may recall that Example 1.4.9 showed $|C_{K,S}| = h_K$, where h_K is the class number of K (i.e the size of the ideal class group). It remains to compute the volume of the second factor.

Note that $\mathcal{O}_K^\times$ can be viewed as the diagonal embedding of $K^\times$ into $\mathbb{I}_{K,S}^1$. By Lemma 4.3.5, we see that

$$\ker(\mathcal{L}_S|_{\mathcal{O}_K^\times}) = \mathcal{O}_{K,\emptyset}^\times = \mu(K),$$

where the second equality follows from the generalized S -unit theorem (Theorem 1.4.3). This gives us the exact sequence

$$1 \rightarrow \mu(K) \rightarrow \mathcal{O}_K^\times \rightarrow \mathcal{L}_S(\mathcal{O}_K^\times) \rightarrow 1. \quad (4.3-4)$$

Since K embeds discretely into any $\mathbb{I}_{K,S}^1$, it follows that $\mathcal{L}_S(\mathcal{O}_K^\times)$ is a discrete subgroup of $\mathcal{L}_S(\mathbb{I}_{K,S}^1) = H_S$, as defined in Lemma 4.3.5. Moreover recall from Lemma 1.4.6 that $\mathbb{I}_{K,S}^1/\mathcal{O}_{K,S}$ is compact, and since $H_S, \mathcal{L}_S(\mathcal{O}_K^\times)$ are the continuous images of $\mathbb{I}_{K,S}^1$ and $\mathcal{O}_{K,S} (= \mathcal{O}_K^\times)$ respectively, we deduce the quotient $H_S/\mathcal{L}_S(\mathcal{O}_K^\times)$ is also compact. This gives us the exact sequence

$$0 \rightarrow \mathcal{L}_S(\mathcal{O}_K^\times) \rightarrow H_S \rightarrow H_S/\mathcal{L}_S(\mathcal{O}_K^\times) \rightarrow 0. \quad (4.3-5)$$

Combining (4.3-4) and (4.3-5) together along with all of our prior observations, we get the following grid of exact sequences:

$$\begin{array}{ccccccc} & & 1 & & 1 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu(K) & \longrightarrow & \mathcal{O}_K^\times (= K \cap \mathbb{I}_{K,S}^1) & \longrightarrow & \mathcal{L}_S(\mathcal{O}_K^\times) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{I}_{K,\emptyset}^1 & \longrightarrow & \mathbb{I}_{K,S}^1 & \longrightarrow & H_S \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{I}_{K,\emptyset}^1/\mu(K) & \longrightarrow & \mathbb{I}_{K,S}^1/(K \cap \mathbb{I}_{K,S}^1) & \longrightarrow & H_S/\mathcal{L}_S(\mathcal{O}_K^\times) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1 & & 1 & & 0 \end{array}$$

It follows that the desired volume we want is given by:

$$\text{Vol}(\mathbb{I}_{K,S}^1/(K \cap \mathbb{I}_{K,S}^1)) = \text{Vol}(H_S/\mathcal{L}_S(\mathcal{O}_K^\times)) \cdot \text{Vol}(\mathbb{I}_{K,\emptyset}^1/\mu(K))$$

Examining the definition of the regulator and the S -idelic logarithm map, we see that $\mathcal{L}_S(\mathcal{O}_K^\times)$ is precisely the embedding of the unit group as a lattice into H_S , and that (the absolute value of) the determinant of the matrix in Definition 4.3.3 is equivalent to the covolume of this lattice. Hence,

$$\text{Vol}(H_S/\mathcal{L}_S(\mathcal{O}_K^\times)) = R_K. \quad (4.3-6)$$

To compute the measure of $\mathbb{I}_{K,\emptyset}^1/\mu(K)$, we use the fact that $\mu(K)$ is a finite subgroup to see that the measure of the quotient is simply given by $\text{Vol}(\mathbb{I}_{K,\emptyset}^1)/|\mu(K)|$. It thus remains to compute this volume. For $v \in S_\infty$, define $U_v := \{\pm 1\}$ if v is real and S^1 if v is complex. Note then that we have

$$\mathbb{I}_{K,\emptyset}^1 = \prod_{v \in S_\infty} U_v \times \prod_{v < \infty} \mathcal{O}_v^\times \subseteq \mathbb{I}_{K,\infty}^1 \subseteq \mathbb{I}_{K,\infty}.$$

First we determine the measure on $\mathbb{I}_{K,\infty}$ (not just for $\mathbb{I}_{K,\infty}^1$!). Recall that our measure we placed on the finite

components of the ideles using the scaling constant of (3.2-1) that gives the unit group volume 1 at all the unramified finite places. Thus if we let T be the set of all infinite and finite ramified (over $\mathbb{Q}$) places, if $d^\times x$ denotes the standard restricted product measure on the ideles, it follows that:

$$d^\times x|_{\mathbb{I}_{K,\infty}} = d^\times x_T = \prod_{v \in T} d^\times x_v.$$

In other words, the Haar measure on $\mathbb{I}_{K,\infty}$ is literally just the product of the normalized measure at all the components in T . At all the finite places ramified over $\mathbb{Q}$, we have

$$\prod_{v < \infty \text{ ramified}} \text{Vol}(\mathcal{O}_v^\times, d^\times x_v) = \prod_{v < \infty \text{ ramified}} N(\mathfrak{p}_v)^{-1/2} = |\mathfrak{d}_K|^{-1/2}.$$

For the infinite places, the Haar measure on U_v should be precisely the one that recovers the measure on $\mathbb{I}_{K,\infty}$ when integrating along $|\mathbb{I}_K|_{\mathbb{I}_K} = \mathbb{R}_{>0}$, in the sense of Fubini. Since the measure on $\mathbb{I}_{K,\infty}$ was seen to be the product of the normalized measures on each component, picking an infinite place $v_0 \in S_\infty$, and noting that $K_v^\times = U_v \times \mathbb{R}_{>0}$, we want to impose the measure on U_v such that

$$\int_{K_v^\times} f(x_v) d^\times x_v = \int_{U_v} \int_{\mathbb{R}_{>0}} f(x_v) \frac{dt}{t} du = \int_{\mathbb{R}_{>0}} \int_{U_v} f(x_v) du \frac{dt}{t}$$

for any $f \in L^1(K_v^\times)$. Since this is the standard Lebesgue measure when $K_v^\times = \mathbb{R}^\times$, we have to give $U_v = \{\pm 1\}$ the counting measure at the real places (we equally add up the volume along the positive and negative half of the real line), and since this is twice the Lebesgue measure when $K_v^\times = \mathbb{C}^\times$, we have to give $U_v = S^1$ measure 2π (the standard Lebesgue measure gives a circle area πr^2). Thus $\text{Vol}(U_v) = 2$ if v is real and 2π if v is complex. Combining these we get

$$\text{Vol}(\mathbb{I}_{K,\emptyset}^1) = \text{Vol}\left(\prod_{v \in S_\infty} U_v \times \prod_{v < \infty \text{ ramified}}\right) = (2)^{r_1} (2\pi)^{r_2} |\mathfrak{d}_K|^{-1/2}. \quad (4.3-7)$$

Putting together our observations and computations in (4.3-3), (4.3-6) and (4.3-7), we get

$$\text{Vol}(C_K^1) = \text{Vol}(\mathbb{I}_{K,S}^1 / (K \cap \mathbb{I}_{K,S}^1)) = \frac{2^{r_1} \cdot (2\pi)^{r_2} h_K R_K}{|\mu(K)| |\mathfrak{d}_K|^{1/2}},$$

as desired. □

We now proceed to prove the corresponding result in the positive characteristic case.

THEOREM 4.3.6. Let K be a global field of positive characteristic with field of constants $\mathbb{F}_q$ and genus g . Then

$$\zeta_K(s) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

where $P \in \mathbb{Z}[t]$ of degree $2g$, with $P(0) = 1$ and $P(1) = |\text{Pic}^0(K)|$.

Proof. Since $|\mathbb{I}_K|_{\mathbb{I}_K} = q^{\mathbb{Z}}$ (by Proposition 4.1.3) and the global L -function in this case takes $|\cdot|_{\mathbb{I}_K}^s$ as an input, we immediately see from Theorem 4.2.4 that $\zeta_K(s)$ may be written as a meromorphic function $f(q^{-s})$, with simple poles at 1 and q^{-1} , corresponding to $s = 0$ and 1 in the proposition statement respectively.

By mimicking our Euler expansion in Example 4.3.2, we see when $K = \mathbb{F}_p(t)$ that we have

$$\zeta_K(s) = \sum_{\text{monic } x \in \mathbb{F}_p[t]} p^{-\deg(x)s} = \sum_{d \geq 0} p^{d(1-s)} = \frac{1}{1 - p^{1-s}}.$$

This is clearly greater than 1 for any s , and for general K of seprable degree n over $\mathbb{F}_p(t)$ we may bound $\zeta_K(s)$ in absolute value from above by $\left| \zeta_{\mathbb{F}_p(t)}(\Re(s)) \right|^n$, for $\Re(s) > 1$. In any event, the expression makes it clear that $\zeta_{\mathbb{F}_p(t)}(s) \rightarrow 1$ as $\Re(s) \rightarrow \infty$, and the same must therefore hold for any K .

Thus, our meromorphic (away from 1 and q^{-1}) function $f(q^{-s})$ tends to 1 as $q^{-s} \rightarrow 0$, and so it must be holomorphic at 0, and be equal to 1 there. From all these and the standard characterization of complex analytic functions we deduce we must have

$$f(x) = \frac{P(x)}{(1-x)(1-qx)},$$

for P an holomorphic function on the complex plane such that $P(0) = 1$.

Recall that the functional equation of the global L -function (Theorem 4.2.5) gives us in this case:

$$\zeta_K(1-s) = \epsilon(s, 1) \zeta_K(s).$$

Letting $x = q^{-s}$, this tells us that

$$P(1/(qx)) = \epsilon(s, 1) P(x)$$

Noting that in this case our multiplicative character $\chi = 1$ is trivial and thus $\text{ord}_v(\chi_v) = 0$ at all v , we recall from the proof of the local functional equation the expression for the local ϵ -factor $\epsilon(1_v)$ (from (2.1-14)), given by

$$\epsilon(s, 1_v) = q^{\text{ord}_v(\psi_v)s} \text{Vol}(\mathfrak{p}_v^{\text{ord}_v(\psi_v)}, dx_v).$$

Here the Haar measure dx_v must be chosen as to be self-dual with respect to our character ψ_v . This is exactly the one that sets the volume of $\mathcal{O}_v$ to be $N(\mathfrak{p}_v^{\text{ord}_v(\psi_v)})^{-1/2} = q^{-\text{ord}_v(\psi_v)/2}$. It thus follows that

$$\epsilon(s, 1_v) = q^{\text{ord}_v(\psi_v)(s-1/2)},$$

and so the global ϵ -factor is given by

$$\epsilon(s, 1) := \prod_v \epsilon(s, 1_v) = q^{(s-1/2)(-\sum_v \text{ord}_v(\psi_v))} = q^{(s-1/2)(2g-2)},$$

using the fact that $\text{div}(\psi)$ is by definition a canonical divisor, and thus has degree $2g - 2$ by our geometric Riemann-Roch theorem. Thus we have:

$$\frac{P(1/(xq))}{(1-q^{s-1})(1-q^s)} = x^{-2g} q^{-g} q^{1-2s} \frac{P(x)}{(1-q^{-s})(1-q^{1-s})}.$$

Multiplying both sides by $1/(q^{1-2s})$ gives us:

$$\begin{aligned} x^{-2g} q^{-g} \frac{P(x)}{(1-q^{-s})(1-q^{1-s})} &= \frac{1}{q^{1-s} \cdot q^{-s}} \frac{P(1/(xq))}{(1-q^{s-1})(1-q^s)} \\ &= \frac{P(1/xq)}{(q^{1-s}-1)(q^{-s}-1)}. \end{aligned}$$

We can cancel out the denominators, since each factor on the right side is off by a factor of -1 , giving us:

$$P(1/(xq)) = x^{-2g} q^{-g} P(x). \quad (4.3-8)$$

Note as $|x| \rightarrow \infty$, we have $P(1/xq) \rightarrow P(0) = 1$, as already shown. Thus we have $|P(x)| = O(|x^{2g}|)$ for x or large enough norm. For any $a \in \mathbb{C}$ of large enough norm, we therefore have $|P(a)| \leq B|a|^{2g}$ for some $B \in \mathbb{R}$. Pick some $R > |a|$, and let C_R denote the circle of radius R centered at the origin of the complex plane. Cauchy

differentiation tells us

$$\begin{aligned} \left| P^{(2g)}(a) \right| &= \left| \frac{(2g)!}{2\pi i} \int_{C_R} \frac{P(z)}{(z-a)^{2g+1}} dz \right| \\ &\leq \frac{(2g)!}{2\pi} \int_0^{2\pi} \frac{BR^{2g}}{R^{2g+1}} d\theta \\ &\leq CB, \end{aligned}$$

for some constant C independent of R . In other words, we have that the $2g$ -th derivative of P is a bounded entire function, hence constant by Liouville's theorem. It follows that P must be a polynomial of degree at most $2g$. Analyzing (4.3-8) as $x \rightarrow 0$ (and recalling that $P(0) = 1 \neq 0$), we see that P must in fact have degree *exactly* $2g$.

Finally, we want to compute $P(1)$. Note that Theorem 4.2.4 tells us that $\zeta_K(s)$ has residue $-\text{Vol}(C_K^1)$ at $s = 0$. Since $(1 - q^{-s})^{-1}$ has residue $\log(q)$ at $s = 0$, we have:

$$-\text{Vol}(C_K^1) = \frac{P(1)}{\log(q)1 - q} \implies P(1) = \text{Vol}(C_K^1)(q - 1) \log(q).$$

Recall from Proposition 3.1.16 that the divisor map gives us the following exact sequence

$$1 \rightarrow \mathbb{I}_{K,\emptyset}^1 / \mathbb{F}_q^\times \rightarrow \mathbb{I}_K^1 / K^\times \rightarrow \text{Pic}^0(K) \rightarrow 0.$$

By similar reasoning as in the number field case (i.e every U_v must be given the measure that recovers the global measure against integration by $q^{\mathbb{Z}}$, in the sense of Fubini), we see $\text{Vol}(\mathbb{I}_{K,\emptyset}^1) = 1/\log(q)$. The exact sequence above then tells us

$$\text{Vol}(C_K^1) = \frac{|\text{Pic}^0(K)|}{\log(q)(q-1)},$$

and combining this with our previous observation immediately yields $P(1) = |\text{Pic}^0(K)|$. □

REMARK. Theorem 4.3.6, in particular the observation that zeta function of a smooth projective curve is rational with specific degree properties related to its genus and reciprocal roots is also linked to the Weil Conjectures, earlier discussed in Example 4.1.2. Our theorem can be seen as a proof of the Weil conjectures (modulo the Riemann hypothesis statement) in the special case of a smooth projective geometrically connected curve.

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