Dinitz's Theorem and Related Results

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1 Introduction

Graph theory is not only tremendously applicable to several real-world applications of modelling, matching and networks, but is also an extremely versatile tool to be wielded by pure mathematicians. In this paper we explore one such application in the combinatorial problem of Latin Squares and a conjecture of Dinitz. The main theorem we show is the following:

Theorem 1.0.1 (Dinitz's Theorem)**.** *Given an $n \times n$ array of cells with a list of n values for each cell, it is always possible to choose a value from each list such that the array forms a Latin Square.*
In graph theory terms, the list chromatic index of $K_{n,n}$ is equal to n.

This is a special case of the result published by Fred Galvin in 1994:

Theorem 1.0.2 (Galvin's Result)**.** *Let G be a bipartite graph, and suppose that $L(G)$ is n-colorable. Then, $L(G)$ is n-choosable.[Gal95]*

The key terms and definitions behind this theorem, along with its relation to the specific Dinitz conjecture, will be discussed in Section 2. The proof of Dinitz's theorem will be given in Section 3. Related concepts and results will be explored in Section 4.

2 Key Ideas and Background

2.1 Latin Squares Problem

Definition. An $n \times n$ **Latin Square** is an array of cells arranged in n rows and n columns populated with n values (often the integers 1 to n) such that no row or column has a repeat value.

It is known that Latin Squares exist for all possible sizes, and we have simple algorithmic ways to construct them. In some sense, the construction and existence of classical Latin Squares are one of the oldest combinatorial problems, with the first results on them being published in 1700 by Korean mathematicians and evidence of them being known as a puzzle as early as 1200 [CD06]. Puzzles like Sudoku and KenKen and board games like Kamisado also draw upon Latin Squares.

What we will be discussing however, is a variation of this old problem that is as follows: suppose that for each cell, you are given a list of n distinct possible values to choose from. No matter what configuration of lists we are given with whatever arbitrary values, is it always possible to create a Latin Square?

Example 2.1.1. *Suppose we have the following grid with the following lists, displayed in Figure 1: From this possible list, we can construct the Latin Square:*

It seems fairly intuitive that, given any set of lists, a construction of a Latin should be possible. After all, the original Latin Square construction is the same problem but with all the lists being $\{1, 2...n\}$. This is what Jeff Dinitz conjectured in 1979 [ERT79], and it remained unsolved for nearly

$$\begin{array}{ccc} \{1,2,3\} & \{1,3,4\} & \{2,5,6\} \\ \{2,3,5\} & \{1,2,3\} & \{4,5,6\} \\ \{4,3,6\} & \{3,5,6\} & \{2,3,5\}, \end{array}$$

Figure 1: A 3x3 set of lists

1	3	5
2	1	6
3	6	2

20 years before Fred Galvin's ingenious solution. To prove this however, and to prove some stronger results, we turn to graph theory.

2.2 Formulation in Graph Theory

The concept of a **proper vertex coloring** of a graph $G = (V, E)$ is well known, where we assign such a color c_i to each vertex v_i such that for all pairs $(v_i, v_j) \in E$, it holds that $c_i \neq c_j$. We now extend this notion to help us solve the conditions of the problem.

Definition. Suppose for every vertex $v_i \in V$, we had an associated list of colors L_i, each of length k. If we can pick $c_i \in L_i$ for all vertices in such a manner that the graph obeys a proper vertex coloring, then we say the graph has a **list coloring** with those lists.
If for every possible combination of lists of length k we can find a list coloring, the graph is said to be k-list-colorable or k**-choosable**. The **choosability** (or **list chromatic number**) of a graph, denoted $ch(G)$, is the smallest k for which it is k-choosable.
The **list chromatic index** or **edge-choosability** refers to the same problem, except with lists for each edge and a proper edge coloring.

Of note here is that being n-choosable is a stronger condition than being n-colorable for a graph. Take the following example of $K_{2,4}$.

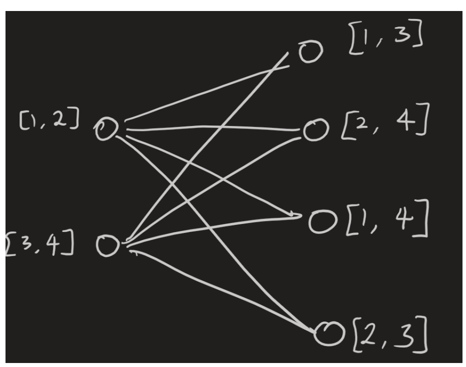


Figure 2: A counter example to show $ch(K_{2,4}) > 2$

Example 2.2.1. *Whatever pair of colors we choose in the left vertex set, there is a vertex in the right vertex set with exactly that list of allowable colors. Thus, this graph is 2-colorable but not 2-choosable.*

To relate it to our problem of List Latin Squares, we want to show that the following graph $\mathcal{G} = (V, E)$ in Figure 3 is n-choosable:

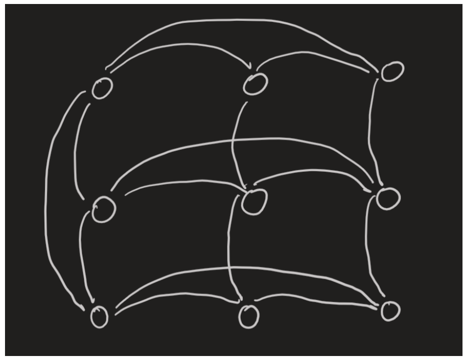


Figure 3: Graph of a 3x3 Latin Square

where $V = [n] \times [n]$, and for all $i \neq j, (v_i \in v_j) \in E$ iff $v_i \cap v_j \neq \emptyset$. In other words, there is an edge between any vertices in the same column or row. This is to ensure they all get different colors, to satisfy the conditions for a Latin square. From here onwards, such a graph will be denoted $\mathcal{G}_n$. Some important properties of this graph are that it has n^2 edges and vertices, and that $\chi(\mathcal{G}_n) = n$, i.e this graph is n-colorable. Thus we know $ch(\mathcal{G}_n) \geq n$, and our goal is to prove equality.

2.3 Line Graphs

In order to translate this graph into a graph with simpler properties and one we are more familiar with, we use line graphs:

Definition. Given a graph $G = (V, E)$, its **line graph** $L(G) = (V', E')$ has vertex set $V' = E$ and $(v_1, v_2) \in E'$ iff $v_1 \cap v_2 \neq \emptyset$.

More intuitively, the edges become vertices, and two vertices in the line graph share an edge if they are incident to a common vertex in the original graph.

Observation. *For any graph,* $G \cong L(L(G))$

In other words, there is a duality between a graph and its line graph. We use this to our advantage in solving the List Latin Squares problem.

Claim. $L(\mathcal{G}_n) \cong K_{n,n}$

Proof. Let the vertices of $K_{n,n}$ be number 1 through n in each independent vertex set V_1, V_2. There is an edge (i, j) for all $i, j \in [n]$. Note that each vertex of $\mathcal{G}_n$ was also associated with an element of $[n] \times [n]$. Moreover, two edges $(i_1, j_1), (i_2, j_2) \in E(K_{n,n})$ are incident to a common vertex iff $i_1 = i_2$

or $j_1 = j_2$. This is also the condition for two vertices of $\mathcal{G}_n$ having an edge between them, and thus we have shown that $L(\mathcal{G}_n)$ is indeed $K_{n,n}$. □

Armed with this claim, we are now ready to understand Theorem 1.0.2 and 1.0.1. We have shown that solving the List Latin Squares problem is equivalent to showing that $\mathcal{G}_n$ is n-choosable, or equivalently, that $K_{n,n}$ is k-edge choosable. 1.0.2 tells us that if $L(K_{n,n}) \cong \mathcal{G}_n$ is n-colorable, which we know it is, then it is n-choosable, which is what we wanted!

And so we have solved the problem. Now the hard part remains: proving Dinitz's theorem.

3 Proving Dinitz's Theorem

3.1 Directed Graphs and Kernels

We begin a discussion of kernels in directed graphs, as they will be a key tool used to prove our main theorem.

Given an undirected graph $G = (V, E)$, we can turn it into a directed graph by taking every edge $(v_1, v_2) \in E$, and turning it into either $(v_1 \to v_2)$ or $(v_2 \to v_1)$. We call a particular configuration of this an **orientation** of G.

Definition. Given directed graph $G = (V, \vec{E})$, a set $K \subseteq V$ is called a **kernel** of G if K is an independent set and for all vertices $w \in V \backslash K$, there exists $v \in K$ such that $(w \to v) \in \vec{E}$.

In other words every vertex not in K must point to a vertex in K. Not every directed graph has a kernel, as shown in the following example:

Example 3.1.1. *Consider the 2 orientations of a graph similar to the house graph depicted in Figure 4:*

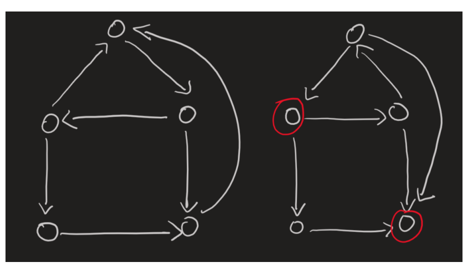


Figure 4

The orientation on the right has a kernel, indicated by the vertices circled in red, but the orientation in the left has no kernel, however hard we may try.

The reason this concept is useful is because of a result due to Bondy, Boppana and Siegel, publish by Noga Alon:

Lemma 3.1.2. *Suppose we have a directed graph $G = (V, \vec{E})$ such that:*

1. *Every induced subgraph of G has a kernel*
2. *For every vertex $v_i \in V$, we have a list L_i of allowable colors satisfying $|L_i| > |\text{outgoing edges from } v_i|$*

Then, G has a list coloring using the lists L_i. [AT92].

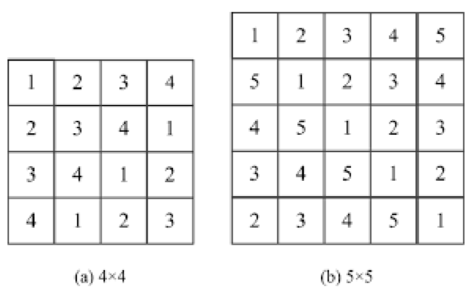
Proof. We use an inductive coloring strategy. We start by picking some color c, and we define $V_C := \{v_i \in V : c \in L_i\}$, and let G_c be the subgraph induced by taking only the vertices in V_c and the edges between them.

By the first condition, we know that G_c has a kernel, let this be denoted K_c. Since K_c is an independent set, we color all of K_c with c. Then, we remove K_c and remove c from all the lists containing c. This means c is removed from the lists of all the vertices in $V_c \backslash K_c$.

Note that all of the remaining vertices in G_c lost exactly outgoing edge, and one color from their list each. Thus, the inequality in condition 2 holds for the rest of the graph. Now, we simply repeat the process with another color. Since at every step for every vertex v_i the condition $|L_i| > |\text{outgoing edges from } v_i|$ holds, we never run out of colors to use. Thus, we continue until the entire graph is colored properly, giving us a list coloring. □

3.2 Kernel Existence

It is an old result that a Latin square can be constructed by taking the set $[n]$ and shifting it horizontally by one cell each row. Two examples are shown in Figure 5



1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

Figure 5: Two Latin Squares. Image from [Cha+19]

We take a Latin square like the one on the right of Figure 5 and construct an orientation on $\mathcal{G}_n$ in the following manner: for entries a, b in the same row or column, we we orient the edge $(a \to b)$ if $a < b$ in the same row or if $a > b$ in the same column. An example is shown in 6.

In this way, a vertex in $\mathcal{G}_n$ with value $k \leq n$ will have $n - k - 1$ outgoing horizontal edges and $k - 1$ outgoing vertical edges, for a total of $n - 1$ outgoing edges.

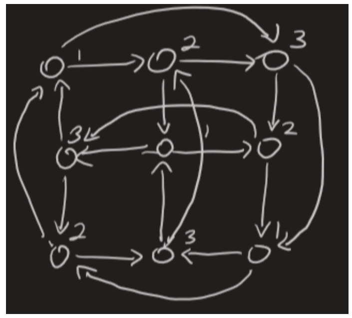


Figure 6: An example on $\mathcal{G}_3$

Now we need to show a kernel exists in $\mathcal{G}_n$. Note that if a kernel does exist, it also exists in every possible subgraph of $\mathcal{G}_n$, since every clique has a kernel. Since every clique is either a subset of a row or a column, we simply pick the kernel to be the vertex with the largest value.

Now we are done, as we have a very simple kernel K we can find in $\mathcal{G}_n$: all of the vertices with value n. These are necessarily an independent set, since they are in a Latin square and thus on separate rows and columns. Moreover, every other vertex is pointing to an element of K, particularly every vertex in every row is pointing to the largest value vertex. We finalise our result in the following:

Proof of Dinitz's Theorem. We have constructed an orientation of $\mathcal{G}_n$ such that each vertex has $n-1$ outgoing edges. We have shown a kernel exists in $\mathcal{G}_n$ such that every subgraph also has a kernel. By Lemma 3.1.2, $\mathcal{G}_n$ has a list coloring with any lists of cardinality greater than $n-1$. In particular, $ch(\mathcal{G}_n) = n$, and we are done. □

4 Beyond Dinitz's Theorem

4.1 Maffray's Theorem, Stable Marriage Theorem

We relate the result we proved about Dinitz's Theorem to a famous theorem on stable matchings of bipartite graphs.

Theorem 4.1.1 (Gale-Shapley's Stable Marriage Theorem)**.** *Given a complete bipartite graph of $K_{n,n}$ with vertex sets A and B where every vertex has a preferential ordering for all the vertices in the other vertex set, it is always possible to find a stable matching [GS62], where a matching is not stable iff:*

1. *There is a vertex in $a \in A$ matched with $b \in B$ that prefers $c \in B$ over b*
2. *The vertex b also prefers some $d \in A$ over a*

Recall that the line graph of $K_{n,n}$ is isomorphic to $\mathcal{G}_n$. An orientation of the graph can be translated into a preferential ordering as follows: if the vertex $(r, c) \in \mathcal{G}_n$ in the rth row and cth column is directed towards (r, c'), i.e $((r,c) \to (r,c')) \in E(\mathcal{G}_n)$, then we say that r prefers c' over c.

Thus, by 4.1.1, we know that a stable matching M must exist on this preferential ordering of $K_{n,n}$. Let $K \subset \mathcal{G}_n$ be the corresponding set of vertices to $M \subset E(K_{n,n})$. We claim K is a kernel of $\mathcal{G}_n$.

Firstly, K is clearly an independent set, by the definition of a matching. Moreover, if $(i,j) \notin M$, then either i prefers someone over j or vice versa, else whatever pair i would be in would be an unstable pair. WLOG we assume $(i,j') \in M$ and i prefers j' over j. Then, it must be that $(i,j) \to (i,j')$ is the orientation of the edge in $\mathcal{G}_n$, which means any vertex not in K points to something in K.

This is can be used to show an important result by Maffray in 1992:

Theorem 4.1.2 (Maffray's Theorem)**.** *We say an orientation of G is **normal** if every clique has a kernel. If a line graph G is perfect, then every normal orientation of G has a kernel [Maf92]. More specifically, every normal orientation of a line graph of a bipartite graph has a kernel.*

That last sentence is given by the fact that every bipartite graph is a subgraph of $K_{n,n}$, for some n. We now have the tools to help us prove Galvin's Result.

4.2 A Broader Result

The main technique is to generalize the idea of a *row* or *column* that we used to show the existence of a normal orientation in $\mathcal{G}_n$.

Proof of 1.0.2. Suppose a bipartite graph $G = (V, E)$ had vertex sets A, B. For all $v \in V$, we call the set $S_v := \{e \in E : v \in e\}$ a **row** if $x \in A$ and a **column** if $x \in B$. Note that each edge is in exactly one row and one column.

We observe that two vertices of the line graph $v_1, v_2 \in L(G) = (V', E')$ are adjacent if they are in the same row or same column. For all $v \in V'$, let $R(v)$ and $C(v)$ denote the row and column containing v.

Consider a proper coloring $f : V' \to [n]$ of $L(G)$ with n colors, each color labelled from 1 to n. Note that this coloring must be one-to-one on each row and column, since all pairs of vertices in $L(G)$ in the same row or column share an edge. Suppose we orient $L(G)$ such that $(v_1 \to v_2) \in E'$ iff either $R(v_1) = R(v_2)$ and $f(v_1) > f(v_2)$ or $C(v_1) = C(v_2)$ and $f(v_1) < f(v_2)$.

Clearly, the number of outgoing vertices from any $v \in V'$ is less than n, since f is one-to-one on $N(v)$. By 3.1.2, all we need to show is that every subgraph of this orientation has a kernel, and we will have shown that it is n-choosable. However, Maffray's result from **??** gives us this for free, since we are considering the line graph of a bipartite graph. Thus, we are done. □

5 Conclusion

In this paper, we have explored results about list colorings of graphs through the lens of the List Latin Squares problem as an introduction. We proved the conjecture of Dinitz, and then proceeded to show a more general result. This work has been further generalized to multigraphs, where we obtain a wider class of choosabilities. This paper was possible due to the papers of the authors referenced throughout this text and the class material of Jacob Fox.

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